

SCALES AND THE FINE STRUCTURE OF $K(\mathbb{R})$ PART III: SCALES OF MINIMAL COMPLEXITY

DANIEL W. CUNNINGHAM

ABSTRACT. We obtain scales of minimal complexity in $K(\mathbb{R})$ using a Levy hierarchy and a fine structure theory for $K(\mathbb{R})$; that is, we identify precisely those levels of the Levy hierarchy for $K(\mathbb{R})$ which possess the scale property.

1. INTRODUCTION

In this paper we shall present a Levy hierarchy for the inner model $K(\mathbb{R})$ and determine the minimal levels of this hierarchy which have the scale property. As a consequence, we shall see that in $K(\mathbb{R})$ there is a close connection between obtaining scales of minimal complexity and new Σ_1 truths about the reals. After we identify the levels of the Levy hierarchy for $K(\mathbb{R})$ which possess the scale property, we will then be able to address the following question, first asked in [1]:

Question (Q). Given an iterable real premouse \mathcal{M} and $n \geq 1$, when does the pointclass $\Sigma_n(\mathcal{M})$ have the scale property?

The boldface pointclass $\Sigma_n(\mathcal{M})$ consists of the sets of reals definable over \mathcal{M} by a Σ_n formula allowing arbitrary constants from the domain of the structure \mathcal{M} to appear in such a definition. More generally, given $X \subseteq M$ the pointclass $\Sigma_n(\mathcal{M}, X)$ consists of the sets of reals definable over \mathcal{M} by a Σ_n formula allowing arbitrary constants from the set X to appear in such a definition. We write $\Sigma_n(\mathcal{M})$ for the pointclass $\Sigma_n(\mathcal{M}, \emptyset)$. Throughout this paper, however, we always allow the set of reals \mathbb{R} to appear as a constant in our relevant languages (see [1, subsections 1.1 & 3.1]). Thus for $n \geq 1$, the pointclass $\Sigma_n(\mathcal{M})$ is equal to the pointclass $\Sigma_n(\mathcal{M}, \{\mathbb{R}\})$; consequently, the pointclass $\Sigma_n(\mathcal{M})$ is not necessarily equal to the pointclass $\Sigma_n(\mathcal{M}, \mathbb{R})$.

In [3] we introduced the Real Core Model $K(\mathbb{R})$ and showed that $K(\mathbb{R})$ is an inner model containing the reals and definable scales beyond those in $L(\mathbb{R})$. To establish our results in [3] on the existence of scales, we defined iterable real premice and extended the *basic* fine structural notions of Dodd-Jensen [6] to encompass iterable “premise above the reals.” Consequently, we were able to prove the following result (see [3, Theorem 4.4]):

Theorem 1.1. *Suppose that \mathcal{M} is an iterable real premouse and that $\mathcal{M} \models \text{AD}$. Then $\Sigma_1(\mathcal{M})$ has the scale property.*

By allowing for real parameters in the proof of Theorem 1.1, we have the following corollary:

Corollary 1.2. *Suppose that \mathcal{M} is an iterable real premouse and that $\mathcal{M} \models \text{AD}$. Then $\Sigma_1(\mathcal{M}, \mathbb{R})$ has the scale property.*

We say that $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ is a real 1-mouse if \mathcal{M} is an iterable real premouse and $\mathcal{P}(\mathbb{R} \times \kappa) \cap \Sigma_1(M) \not\subseteq M$, where M has the form $J_\alpha[\mu](\mathbb{R})$ and κ is the “measurable cardinal” in \mathcal{M} (see [1, subsections 3.2 & 3.3]). Real 1-mice suffice to define the real core model and to prove the results in [3] about $K(\mathbb{R})$; however, real 1-mice are not sufficient to construct scales of minimal complexity in $K(\mathbb{R})$. Our solution to the problem of identifying these scales requires the development of a *full* fine structure theory for $K(\mathbb{R})$. In the paper [4] we initiated this development by generalizing Dodd-Jensen’s notion of a mouse to that of a *real mouse* (see [1, subsection 3.4]). This is accomplished by replacing Σ_1 with Σ_n , where n is the smallest integer such that $\mathcal{P}(\mathbb{R} \times \kappa) \cap \Sigma_{n+1}(\mathcal{M}) \not\subseteq M$, together with a stronger iterability condition.

2000 *Mathematics Subject Classification.* Primary: 03E15; Secondary: 03E45, 03E60.
Key words and phrases. Descriptive set theory, scales, determinacy, fine structure.

Let \mathcal{M} be a real mouse and assume that there is an integer $m \geq 1$ such that $\mathcal{P}(\mathbb{R}) \cap \Sigma_m(\mathcal{M}) \not\subseteq M$. We shall let $\mathfrak{m} = m(\mathcal{M})$ denote the least such integer and, in this case, we say that \mathcal{M} is *weak* if

- (1) \mathcal{M} is a proper initial segment of an iterable real premouse, and
- (2) \mathcal{M} realizes a $\Sigma_{\mathfrak{m}}$ type not realized in any proper initial segment of \mathcal{M} .

In (2) a $\Sigma_{\mathfrak{m}}$ type is a non-empty subset Σ of

$$\{\theta \in \Sigma_{\mathfrak{m}} \cup \Pi_{\mathfrak{m}} : \theta \text{ is a formula of one free variable}\}$$

and \mathcal{M} is said to realize Σ if there is an $a \in M$ such that $\mathcal{M} \models \theta(a)$ for all $\theta \in \Sigma$. Finally, if \mathcal{M} satisfies (1) but fails to satisfy (2), then we say that \mathcal{M} is *strong*.

Using the fine structure of real mice developed in [4] and [1], we establish in Part II [2, Theorem 6.1] the following theorem on the existence of scales:

Theorem 1.3. *Suppose that \mathcal{M} is a weak real mouse satisfying AD. Then $\Sigma_{\mathfrak{m}}(\mathcal{M})$ has the scale property, where $\mathfrak{m} = m(\mathcal{M})$.*

Remark 1.4. We specifically note that the proofs of both Theorem 1.1 and Theorem 1.3 require *only* the determinacy of sets of reals in the relevant \mathcal{M} . This is a critical property for proving that $K(\mathbb{R})$ satisfies AD under certain hypotheses.

Theorem 1.3 is an essential component in our analysis of scales in $K(\mathbb{R})$. Our next result, which follows from the proof of Theorem 1.3 (see [2, Theorem 6.4]), is also an important ingredient in our analysis of scales in $K(\mathbb{R})$.

Theorem 1.5. *Suppose that \mathcal{M} is a weak real mouse and let $\mathfrak{m} = m(\mathcal{M})$. For any set of reals $P \in \Sigma_{\mathfrak{m}}(\mathcal{M})$, there exists a total $\Sigma_{\mathfrak{m}}(\mathcal{M})$ map $k: \omega \rightarrow M$ such that $P = \bigcup_{i \in \omega} k(i)$.*

Theorems 1.1 and 1.3 are the key results that we will use in this paper to give a complete description of those levels of the Levy hierarchy for $K(\mathbb{R})$ which have the scale property. The work presented here can be viewed as a generalization of Steel's work on the existence of scales in the inner model $L(\mathbb{R})$. Steel [12] develops a fine structure theory and a Levy hierarchy for $L(\mathbb{R})$. Using this development, Steel solves the problem of finding scales of minimal complexity in $L(\mathbb{R})$ and, as a consequence, shows that there is a close connection between obtaining such scales and new Σ_1 truths in $L(\mathbb{R})$ about the reals.

Our paper is organized into 7 sections. In Section 2 we review the definitions of $L(\mathbb{R})$ and $K(\mathbb{R})$. Section 3 discusses the notion of a Σ_1 -gap. Section 4 presents a complete description of those levels of the Levy hierarchy for iterable real premice which have the scale property. Section 5 shows that the premouse iteration of a premouse preserves its Σ_1 -gaps and preserves its internal pointclasses (see Definition 3.6). We define a Levy hierarchy for $K(\mathbb{R})$ in Section 6 and then construct scales of minimal complexity in $K(\mathbb{R})$. In Section 7 we present an answer to the question posed at the beginning of this paper.

1.1. Preliminaries and notation. Let ω be the set of all natural numbers. $\mathbb{R} = {}^\omega\omega$ is the set of all functions from ω to ω . We call \mathbb{R} the set of reals and regard \mathbb{R} as a topological space by giving it the product topology, using the discrete topology on ω . For a set $A \subseteq \mathbb{R}$ we associate a two person infinite game on ω , with *payoff* A , denoted by G_A :

$$\begin{array}{ccccccc} \text{I} & & x(0) & & x(2) & & \\ & & & & & & \\ \text{II} & & & & x(1) & & x(3) \quad \dots \end{array}$$

in which player **I** wins if $x \in A$, and **II** wins if $x \notin A$. We say that A is *determined* if the corresponding game G_A is determined, that is, either player **I** or **II** has a winning strategy (see [11, page 287]). The *axiom of determinacy* (AD) is a regularity hypothesis about games on ω and states: $\forall A \subseteq \mathbb{R}$ (A is determined).

We work in $\text{ZF} + \text{DC}$ and state any additional hypotheses as we need them, to keep a close watch on the use of determinacy in the proofs of our main theorems. Variables $x, y, z, w \dots$ generally range over \mathbb{R} , while $\alpha, \beta, \gamma, \delta \dots$ range over OR , the class of ordinals. The cardinal Θ is the supremum of the ordinals which are the surjective image of \mathbb{R} .

A *pointclass* is a set of subsets of \mathbb{R} closed under recursive substitutions. A **boldface pointclass** is a pointclass closed under continuous substitutions. For a pointclass Γ , one usually writes “ Γ -AD” or “ $\text{Det}(\Gamma)$ ” to denote the assertion that all games on ω with payoff in Γ are determined. For the notions of a scale and of the scale property (and any other notions from Descriptive Set Theory which we have not defined), we refer the reader to Moschovakis [11].

A proper class M is called an *inner model* if and only if M is a transitive \in -model of ZF containing all the ordinals. For an inner model M with $X \in M$ we shall write $\mathcal{P}^M(X)$ to denote the power set of X as computed in M . For an ordinal $\kappa \in M$, we shall abuse standard notation slightly and write ${}^\kappa M = \{f \in M \mid f: \kappa \rightarrow M\}$.

We distinguish between the notations $L[A]$ and $L(A)$. The inner model $L(A)$ is defined to be the class of sets constructible *above* A , that is, one starts with a set A and iterates definability in the language of set theory. Thus, $L(A)$ is the smallest inner model M such that $A \in M$. The inner model $L[A]$ is defined to be the class of sets constructible *relative* to A , that is, one starts with the empty set and iterates definability in the language of set theory augmented by the predicate A . Consequently, $L[A]$ is the smallest inner model M such that $A \cap M \in M$ (see page 34 of [9]). Furthermore, one defines $L[A, B]$ to be the class of sets constructible *relative* to A and B , whereas $L[A](B)$ is defined as the class of sets constructible *relative* to A and *above* B . Thus, $A \cap L[A](B) \in L[A](B)$ and $B \in L[A](B)$.

Given a model $\mathcal{M} = (M, c_1, c_2, \dots, c_m, A_1, A_2, \dots, A_N)$, where the A_i are predicates and the c_i are constants, if $X \subseteq M$ then $\Sigma_n(\mathcal{M}, X)$ is the class of relations on M definable over \mathcal{M} by a Σ_n formula from parameters in $X \cup \{c_1, c_2, \dots, c_m\}$. $\Sigma_\omega(\mathcal{M}, X) = \bigcup_{n \in \omega} \Sigma_n(\mathcal{M}, X)$. We write “ $\Sigma_n(\mathcal{M})$ ” for $\Sigma_n(\mathcal{M}, \emptyset)$ and “ $\mathbf{\Sigma}_n(\mathcal{M})$ ” for the boldface class $\Sigma_n(\mathcal{M}, M)$. Similar conventions hold for Π_n and Δ_n notations. If \mathcal{M} is a substructure of \mathcal{N} and $X \subseteq M \subseteq N$, then “ $\mathcal{M} \prec_n^X \mathcal{N}$ ” means that $\mathcal{M} \models \phi[a]$ if and only if $\mathcal{N} \models \phi[a]$, for all $a \in (X)^{<\omega}$ and for all Σ_n formulae ϕ (the formula ϕ is allowed constants taken from $\{c_1, c_2, \dots, c_m\}$). We write “ $\mathcal{M} \prec_n \mathcal{N}$ ” for “ $\mathcal{M} \prec_n^M \mathcal{N}$.” In addition, for any two models \mathcal{M} and \mathcal{N} , we write $\pi: \mathcal{M} \xrightarrow{\Sigma_n} \mathcal{N}$ to indicate that the map π is a Σ_n -elementary embedding, that is, $\mathcal{M} \models \phi[a]$ if and only if $\mathcal{N} \models \phi[\pi(a)]$, for all $a = \langle a_0, a_1, \dots \rangle \in (M)^{<\omega}$ and for all Σ_n formulae ϕ , where $0 \leq n \leq \omega$ and $\pi(a) = \langle \pi(a_0), \pi(a_1), \dots \rangle$.

We now give a brief definition of an iterable real premice (for more details see [1] or [3]). First, we present a preliminary definition.

Definition 1.6. Let μ be a normal measure on κ . We say that μ is an \mathbb{R} -complete measure on κ if the following holds: if $\langle A_x : x \in \mathbb{R} \rangle$ is any sequence such that $A_x \in \mu$ for all $x \in \mathbb{R}$, then $\bigcap_{x \in \mathbb{R}} A_x \in \mu$.

For $N \in \omega$, the language

$$\mathcal{L}_N = \{\in, \mathbb{R}, \underline{\kappa}, \mu, A_1, \dots, A_N\}$$

consists of the constant symbols \mathbb{R} and $\underline{\kappa}$ together with the membership relation \in and the predicate symbols μ, A_1, \dots, A_N .

Definition 1.7. A model $\mathcal{M} = (M, \in, \mathbb{R}^{\mathcal{M}}, \underline{\kappa}^{\mathcal{M}}, \mu, A_1, \dots, A_N)$ is a *premouse* (above the reals) if

- (1) \mathcal{M} is a transitive set model of $V = L[\mu, A_1, \dots, A_N](\mathbb{R})$
- (2) $\mathcal{M} \models “\mu \text{ is an } \mathbb{R}\text{-complete measure on } \underline{\kappa}”$.

\mathcal{M} is a *pure premouse* if $\mathcal{M} = (M, \mathbb{R}^{\mathcal{M}}, \underline{\kappa}^{\mathcal{M}}, \mu)$ (that is, $N = 0$). Finally, \mathcal{M} is a *real premouse* if it is pure and $\mathbb{R}^{\mathcal{M}} = \mathbb{R}$.

A real premouse \mathcal{M} has a natural Jensen hierarchy. For any $\alpha \in \text{OR}^{\mathcal{M}}$ we let $S_\alpha^{\mathcal{M}}(\mathbb{R})$ denote the unique set in \mathcal{M} satisfying $\mathcal{M} \models \exists f(\varphi(f) \wedge \alpha \in \text{dom}(f) \wedge S_\alpha^{\mathcal{M}}(\mathbb{R}) = f(\alpha))$, where φ is the Σ_0 sentence used to define the sequence $\langle S_\gamma^{\mathcal{M}}(\mathbb{R}) : \gamma < \text{OR}^{\mathcal{M}} \rangle$ (see Definition 1.5 of [3]). For $\lambda = \text{OR}^{\mathcal{M}}$, let $S_\lambda^{\mathcal{M}}(\mathbb{R}) = \bigcup_{\alpha < \lambda} S_\alpha^{\mathcal{M}}(\mathbb{R})$. Let $\widehat{\text{OR}}$ denote the class of ordinals

$\{\gamma : \text{the ordinal } \omega\gamma \text{ exists}\}$ and let $J_\gamma^{\mathcal{M}}(\mathbb{R}) = S_{\omega\gamma}^{\mathcal{M}}(\mathbb{R})$, for $\gamma \leq \widehat{\text{OR}}^{\mathcal{M}}$. It follows that $M = J_\alpha^{\mathcal{M}}(\mathbb{R})$ where $\alpha = \widehat{\text{OR}}^{\mathcal{M}}$. Let \mathcal{M}^γ be the substructure of \mathcal{M} defined by $\mathcal{M}^\gamma = (J_\gamma^{\mathcal{M}}(\mathbb{R}), \in, \mathbb{R}, J_\gamma^{\mathcal{M}}(\mathbb{R}) \cap \mu)$

for $1 < \gamma \leq \widehat{\text{OR}}^{\mathcal{M}}$, and let $M^\gamma = J_\gamma^{\mathcal{M}}(\mathbb{R})$. We can write $\mathcal{M}^\gamma = (M^\gamma, \in, \mathbb{R}, \mu)$, as this will cause no confusion. In particular, \mathcal{M}^γ is *amenable*, that is, $a \cap \mu \in M^\gamma$, for all $a \in M^\gamma$.

Given a premouse \mathcal{M} we can construct iterated ultrapowers and obtain a commutative system of models by taking direct limits at limit ordinals.

Definition 1.8. Let \mathcal{M} be a premouse. Then

$$(1.1) \quad \langle \langle \mathcal{M}_\gamma \rangle_{\gamma \in \text{OR}}, \langle \pi_\gamma : \mathcal{M} \xrightarrow[\Sigma_1]{} \mathcal{M}_\gamma \rangle_{\gamma \in \text{OR}} \rangle$$

is the commutative system satisfying the inductive definition:

- (1) $\mathcal{M}_0 = \mathcal{M}$
- (2) $\pi_{\gamma\gamma} = \text{identity map}$, and $\pi_{\beta\gamma} \circ \pi_{\alpha\beta} = \pi_{\alpha\gamma}$ for all $\alpha \leq \beta \leq \gamma \leq \lambda$
- (3) If $\lambda = \lambda' + 1$, then $\mathcal{M}_\lambda = \text{ultrapower of } \mathcal{M}_{\lambda'}$, and $\pi_{\alpha\lambda} = \pi^{\mathcal{M}_{\lambda'}} \circ \pi_{\alpha\lambda'}$ for all $\alpha \leq \lambda'$
- (4) If λ is a limit ordinal, then $\langle \mathcal{M}_\lambda, \langle \pi_{\alpha\lambda} : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\lambda \rangle_{\alpha < \lambda} \rangle$ is the direct limit of

$$\left\langle \langle \mathcal{M}_\alpha \rangle_{\alpha < \lambda}, \langle \pi_{\alpha\beta} : \mathcal{M}_\alpha \rightarrow \mathcal{M}_\beta \rangle_{\alpha \leq \beta < \lambda} \right\rangle.$$

The commutative system in the above (1.1) is called the *premouse iteration* of \mathcal{M} . We note that the maps in the above commutative system are cofinal and are Σ_1 embeddings, that is,

$$\pi_{\alpha\beta} : \mathcal{M}_\alpha \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{M}_\beta$$

for all $\alpha \leq \beta \in \text{OR}$. We shall call $\pi_{0\beta} : \mathcal{M} \xrightarrow[\Sigma_1]{} \mathcal{M}_\beta$ the *premouse embedding* of \mathcal{M} into its β^{th} *premouse iterate* \mathcal{M}_β .

Definition 1.9. A premouse \mathcal{M} is an *iterable premouse* if \mathcal{M}_λ is well-founded for all $\lambda \in \text{OR}$.

For an iterable premouse \mathcal{M} and $\alpha \in \text{OR}$, we identify \mathcal{M}_α with its transitive collapse. It follows that

$$\mathcal{M}_\alpha = (M_\alpha, \in, \mathbb{R}^{\mathcal{M}_\alpha}, \underline{\kappa}^{\mathcal{M}_\alpha}, \mu, A_1, \dots, A_N)$$

is a premouse and we write $\kappa_\alpha = \pi_{0\alpha}(\underline{\kappa}^{\mathcal{M}}) = \underline{\kappa}^{\mathcal{M}_\alpha}$ for $\alpha \in \text{OR}$.

Whenever we write $\underline{\Sigma}_n(\mathcal{M}) = \underline{\Sigma}_n(\mathcal{N})$, we implicitly mean that this is an equality between pointclasses.

Definition 1.10. Let \mathcal{M} and \mathcal{N} be real premice. We shall say that $\underline{\Sigma}_n(\mathcal{M}) = \underline{\Sigma}_n(\mathcal{N})$ as *pointclasses* if for every Σ_n formula of one variable $\varphi(v)$ with constants from \mathcal{M} , there is a Σ_n formula of one variable $\psi(v)$ with constants from \mathcal{N} such that $\mathcal{M} \models \varphi(x)$ if and only if $\mathcal{N} \models \psi(x)$ for all $x \in \mathbb{R}$, and vice versa.

We now recall the definition of Σ_ω^μ formulae and some other notions from [1]. A real premouse $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is a model of the language $\mathcal{L} = \{\in, \mathbb{R}, \underline{\kappa}, \mu\}$ where μ is a predicate. At times, we will want to add a quantifier to the language \mathcal{L} . Since the quantifier extends the predicate μ in our intended structures, we shall use the same symbol μ for this quantifier. We shall denote this expanded language by \mathcal{L}^μ and write Σ_ω^μ for the formulae in this expanded language. For γ such that $\kappa < \gamma < \widehat{\text{OR}}^{\mathcal{N}}$, let $\mu^{\gamma+1} = N^{\gamma+1} \cap \mu$. Then $(\mathcal{N}^\gamma, \mu^{\gamma+1})$ is an \mathcal{L}^μ structure, where the new quantifier symbol is to be interpreted by $\mu^{\gamma+1}$. That is, $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \models (\mu \varkappa \in \kappa) \psi(\alpha)$ if and only if $\{\varkappa \in \kappa : (\mathcal{N}^\gamma, \mu^{\gamma+1}) \models \psi(\varkappa)\} \in \mu^{\gamma+1}$. The following is Definition 3.86 of [1], but with an additional clause.

Definition 1.11. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be a real premouse and let $\kappa < \gamma < \widehat{\text{OR}}^{\mathcal{N}}$. We shall say that $\mu^{\gamma+1}$ is *predictable* if the following holds: For each \mathcal{L} formula $\chi(v_0, v_1, \dots, v_k)$ there is another \mathcal{L} formula $\psi(v_1, \dots, v_k)$ with a parameter $d \in N^\gamma$ such that for all $a_1, \dots, a_k \in N^\gamma$

$$(1.2) \quad B_{a_1, \dots, a_k} \in \mu^{\gamma+1} \text{ iff } \mathcal{N}^\gamma \models \psi(a_1, \dots, a_k)$$

where $B_{a_1, \dots, a_k} = \{\varkappa \in \kappa : \mathcal{N}^\gamma \models \chi(\varkappa, a_1, \dots, a_k)\}$. If there is single parameter $d \in N^\gamma$ that satisfies (1.2) for all such formula ψ , then we will state that $\mu^{\gamma+1}$ is *d-predictable*.

We will now go over the definition of a real 1-mouse.

Definition 1.12. Let \mathcal{M} be an iterable real premouse. The *projectum* $\rho_{\mathcal{M}}$ is the least ordinal $\rho \leq \widehat{\text{OR}}^{\mathcal{M}}$ such that $\mathcal{P}(\mathbb{R} \times \omega\rho) \cap \Sigma_1(\mathcal{M}) \not\subseteq M$, and $p_{\mathcal{M}}$ is the \leq_{BK} -least $p \in [\text{OR}^{\mathcal{M}}]^{<\omega}$ such that $\mathcal{P}(\mathbb{R} \times \omega\rho_{\mathcal{M}}) \cap \Sigma_1(\mathcal{M}, \{p\}) \not\subseteq M$.

Definition 1.13. An iterable real premouse \mathcal{M} is a *real 1-mouse* if $\omega\rho_{\mathcal{M}} \leq \kappa^{\mathcal{M}}$.

Real 1-mice suffice to define the class $K(\mathbb{R})$ and to prove the results in [3]; however, real 1-mice are not sufficient to construct scales of minimal complexity. Our solution to the problem of identifying these scales in $K(\mathbb{R})$ requires the development of a *full* fine structure theory for $K(\mathbb{R})$. In the paper [4] we initiated this development by generalizing Dodd-Jensen's notion of a mouse to that of a *real mouse* (see subsection 3.4 of [1]). This is accomplished by

- isolating the concept of *acceptability above the reals*¹ (see [1, Definition 3.15]),
- replacing Σ_1 with Σ_n , where n is the smallest integer such that $\mathcal{P}(\mathbb{R} \times \kappa) \cap \Sigma_{n+1}(\mathcal{M}) \not\subseteq M$,
- defining an iteration procedure stronger than premouse iteration.

Let $\mathcal{M} = (M, \mathbb{R}, \underline{\kappa}^{\mathcal{M}}, \mu)$ be an iterable real premouse. The Σ_1 -master code $A_{\mathcal{M}}$ of \mathcal{M} is the set

$$A_{\mathcal{M}} = \{(x, s) \in \mathbb{R} \times (\omega\rho_{\mathcal{M}})^{<\omega} : \mathcal{M} \models \varphi_{x(0)}[\lambda n.x(n+1), s, p_{\mathcal{M}}]\}$$

where $\langle \varphi_i : i \in \omega \rangle$ is a fixed recursive listing of all the Σ_1 formulae of three variables in the language $\mathcal{L} = \{\in, \mathbb{R}, \underline{\kappa}, \mu\}$.

Theorem 4.1 of [1] proves that an iterable real premouse \mathcal{M} is acceptable above the reals. Using $A_{\mathcal{M}}$ and $\omega\rho_{\mathcal{M}}$ one defines a new structure with domain $H_{\omega\rho_{\mathcal{M}}}^{\mathcal{M}} = \{a \in M : |T_c(a)|_{\mathcal{M}} < \omega\rho_{\mathcal{M}}\}$ where $T_c(a)$ denotes “the transitive closure of a ” and $|a|_{\mathcal{M}}$ denotes the least ordinal λ in \mathcal{M} such that $f : \lambda \times \mathbb{R} \xrightarrow{\text{onto}} a$ for some $f \in M$. Let $M^1 = H_{\omega\rho_{\mathcal{M}}}^{\mathcal{M}}$. The Σ_1 -code of \mathcal{M} is the structure $\mathcal{M}^1 = (M^1, \mathbb{R}, \underline{\kappa}^{\mathcal{M}}, \mu, A_1)$, where $A_1 = A_{\mathcal{M}}$. Because \mathcal{M} is acceptable, we can repeat this construction and inductively define structures \mathcal{M}^n in the language $\mathcal{L}_n = \{\in, \mathbb{R}, \underline{\kappa}, \mu, A_1, \dots, A_n\}$ where the predicate symbols A_1, \dots, A_n represent the previously defined master codes and thus, one can define $\rho_{\mathcal{M}}^n$ and $A_{\mathcal{M}^n}$. When there is an integer n such that $\rho_{\mathcal{M}}^{n+1} \leq \kappa^{\mathcal{M}} < \rho_{\mathcal{M}}^n$, then we say that \mathcal{M} is *critical* and we let $n = n(\mathcal{M})$ denote this integer. If the structure \mathcal{M}^n is sufficiently iterable, then we say that \mathcal{M} is a *real mouse*. More specifically, let $\overline{\mathcal{M}} = \mathcal{M}^n$. Since $\overline{\mathcal{M}}$ is an iterable real premouse, let

$$(1.3) \quad \langle \langle \overline{\mathcal{M}}_{\alpha} \rangle_{\alpha \in \text{OR}}, \langle \overline{\pi}_{\alpha\beta} : \overline{\mathcal{M}}_{\alpha} \xrightarrow[\Sigma_1]{\text{cofinal}} \overline{\mathcal{M}}_{\beta} \rangle_{\alpha \leq \beta \in \text{OR}} \rangle$$

be the *premouse iteration* of $\overline{\mathcal{M}}$ as in Definition 1.8. We can extend the system (1.3) of transitive models via the extension of embeddings lemma (Lemma 3.64 of [1]) and obtain the commutative system of transitive structures

$$(1.4) \quad \langle \langle \mathcal{M}_{\alpha} \rangle_{\alpha \in \text{OR}}, \langle \pi_{\alpha\beta} : \mathcal{M}_{\alpha} \xrightarrow[\Sigma_{n+1}]{} \mathcal{M}_{\beta} \rangle_{\alpha \leq \beta \in \text{OR}} \rangle.$$

The system (1.4) is called the *mouse iteration* of \mathcal{M} . We shall call $\pi_{0\beta} : \mathcal{M} \xrightarrow[\Sigma_{n+1}]{} \mathcal{M}_{\beta}$ the *mouse embedding* of \mathcal{M} into its β^{th} *mouse iterate* \mathcal{M}_{β} .

Remark. A real 1-mouse $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ is the simplest of real mice; because \mathcal{M} is iterable and $\mathcal{P}(\mathbb{R} \times \kappa) \cap \Sigma_1(M) \not\subseteq M$.

We now review the definition of the core of a real mouse \mathcal{M} . Let $\overline{\mathcal{M}} = \mathcal{M}^n$, where $n = n(\mathcal{M})$, and let $\mathcal{H} = \text{Hull}_1^{\overline{\mathcal{M}}}(\mathbb{R} \cup \omega\rho_{\overline{\mathcal{M}}} \cup \{p_{\overline{\mathcal{M}}}\})$. Thus, $\mathcal{H} \prec_1 \overline{\mathcal{M}}$. Let $\overline{\mathcal{C}}$ be the transitive collapse of \mathcal{H} . By Lemma 3.64 of [1] there is a decoding \mathcal{C} of $\overline{\mathcal{C}}$ and a map $\sigma : \mathcal{C} \xrightarrow[\Sigma_{n+1}]{} \mathcal{M}$. It follows that \mathcal{C} is a real mouse with $n(\mathcal{C}) = n(\mathcal{M})$. We denote \mathcal{C} by $\mathcal{C}(\mathcal{M})$. Let

$$\langle \langle \overline{\mathcal{C}}_{\alpha} \rangle_{\alpha \in \text{OR}}, \langle \overline{\pi}_{\alpha\beta} : \overline{\mathcal{C}}_{\alpha} \xrightarrow[\Sigma_1]{} \overline{\mathcal{C}}_{\beta} \rangle_{\alpha \leq \beta \in \text{OR}} \rangle$$

¹This concept extends the Dodd-Jensen notion of acceptability to include the set of reals.

be the premouse iteration of $\overline{\mathfrak{C}}$ and let

$$\langle \langle \mathfrak{C}_\alpha \rangle_{\alpha \in \text{OR}}, \langle \pi_{\alpha\beta} : \mathfrak{C}_\alpha \xrightarrow{\Sigma_{n+1}} \mathfrak{C}_\beta \rangle_{\alpha \leq \beta \in \text{OR}} \rangle$$

be the mouse iteration of \mathfrak{C} . It follows that \mathcal{M} is a mouse iterate of \mathfrak{C} ; that is, there is an ordinal θ such that $\overline{\mathfrak{C}}_\theta = \overline{\mathcal{M}}$ and $\mathfrak{C}_\theta = \mathcal{M}$.

Let $\kappa_\alpha = \pi_{0\alpha}(\kappa^\mathfrak{C})$ for $\alpha \in \text{OR}$. In particular, $\kappa_0 = \kappa^\mathfrak{C}$. Given that $\mathfrak{C}_\theta = \mathcal{M}$, let $I_m = \{ \kappa_\alpha : \alpha \text{ is } m\text{-good} \wedge \alpha < \theta \}$, where an ordinal α is m -good if and only if α is a multiple of ω^m . Corollary 3.81 of [1] asserts that I_m is a set of order $\Sigma_{m+1}(\mathcal{M}, \{ \pi_{0\theta}(a) : a \in C \})$ indiscernibles, where C is the domain of \mathfrak{C} . We now state a result from [1] that we use in subsection 4.3 when we deal with the existence of scales at the “end of a gap.”

Lemma 1.14. *Let \mathcal{M} be a mouse with core \mathfrak{C} and let $\mathfrak{n} = n(\mathcal{M})$. Suppose that $\rho_{\mathcal{M}}^{\mathfrak{n}+1} < \kappa^\mathcal{M}$ and that $\mathcal{M} = \mathcal{N}^\gamma$ for an iterable premouse \mathcal{N} where $\kappa^\mathcal{M} = \kappa^\mathcal{N} = \kappa$ $\kappa < \gamma < \widehat{\text{OR}}^\mathcal{N}$. Let θ be such that the mouse iterate $\mathfrak{C}_\theta = \mathcal{M}$ and let $\kappa_0 = \kappa^\mathfrak{C}$. Then*

- (1) $\theta = \kappa$, θ is a multiple of ω^ω and $I_m \in \mu^\mathcal{N}$ for all $m \in \omega$
- (2) I_m is uniformly $\Sigma_\omega(\overline{\mathcal{M}}, \{ \kappa_0 \})$
- (3) I_m is uniformly $\Sigma_\omega(\mathcal{M}, \{ \kappa_0 \})$ and its definition depends only on \mathfrak{n}
- (4) $\mu^{\gamma+1}$ is κ_0 -predictable.

Proof. Items (1) and (2) of the above list follow directly from Lemma 3.83 and Lemma 3.88 in [1]. Since each I_m is uniformly $\Sigma_\omega(\overline{\mathcal{M}}, \{ \kappa_0 \})$ and because $\overline{\mathcal{M}}$ is definable over \mathcal{M} with a definition depending only on \mathfrak{n} , we see that (3) holds. Our proof of Lemma 3.88 in [1] assumes that I_m is $\Sigma_\omega(\mathcal{M})$ and proves, as a claim, that $\mu^{\gamma+1}$ is predictable. However, since each I_m is $\Sigma_\omega(\mathcal{M}, \{ \kappa_0 \})$, our proof of Lemma 3.88 is easily modified to show that $\mu^{\gamma+1}$ is κ_0 -predictable. \square

Recall that a real premouse \mathcal{M} is a pure premouse; that is, it has the form $\mathcal{M} = (M, \mathbb{R}, \underline{\kappa}^\mathcal{M}, \mu)$. For the remainder of this paper, when we say that a structure \mathcal{N} is a premouse we shall mean, for the most part, that \mathcal{N} is real premouse. It will be clear from the context when we are actually working with premice that are not pure. For more details on the matters discussed in this section, see [1, Section 3].

2. THE REAL CORE MODEL

In this section we shall review the basic definitions of $L(\mathbb{R})$ and $K(\mathbb{R})$.

2.1. The inner model $L(\mathbb{R})$. We review the Jensen hierarchy for $L(\mathbb{R})$. We presume the reader is familiar with the rudimentary functions (see [8]). Let $\text{rud}(M)$ be the closure of $M \cup \{M\}$ under the rudimentary functions. Let

$$\begin{aligned} J_1(\mathbb{R}) &= V_{\omega+1} = \{a : \text{rank}(a) \leq \omega\}, \\ J_{\alpha+1}(\mathbb{R}) &= \text{rud}(J_\alpha(\mathbb{R})) \text{ for } \alpha > 0, \\ J_\lambda(\mathbb{R}) &= \bigcup_{\alpha < \lambda} J_\alpha(\mathbb{R}) \text{ for limit ordinals } \lambda. \end{aligned}$$

$L(\mathbb{R}) = \bigcup_{\alpha \in \text{OR}} J_\alpha(\mathbb{R})$ is the smallest inner model of ZF containing the reals. Under the hypothesis that $L(\mathbb{R})$ is a model of AD+DC, researchers have essentially settled all the important problems concerning the descriptive set theory and structure of $L(\mathbb{R})$.

2.2. The inner model $K(\mathbb{R})$. Using real 1-mice there is a natural way to define an inner model larger than $L(\mathbb{R})$.

Definition 2.1. The real core model is the class $K(\mathbb{R}) = \{x : \exists \mathcal{N} (\mathcal{N} \text{ is a real 1-mouse} \wedge x \in \mathcal{N})\}$.

One can prove that $K(\mathbb{R})$ is an inner model of ZF and contains a set of reals not in $L(\mathbb{R})$ (see [3]). It turns out that problems concerning the descriptive set theory and structure of $K(\mathbb{R})$ can also be settled under the hypothesis that $K(\mathbb{R})$ is a model of AD. For example, using a mixture of descriptive set theory, fine structure and the theory of iterated ultrapowers, one can produce definable scales in $K(\mathbb{R})$ beyond those in $L(\mathbb{R})$ and prove that $K(\mathbb{R}) \models \text{DC}$.

Remark 2.2. There exists an iterable real premouse if and only if $\mathbb{R}^\#$ exists. So $K(\mathbb{R})$ is nonempty if and only if $\mathbb{R}^\#$ exists. Therefore, we will implicitly assume that $\mathbb{R}^\#$ exists.

3. Σ_1 -GAPS

Let M be transitive model above the reals with a “cumulative” hierarchy, say $M = \bigcup_{\alpha \in \text{OR}} M_\alpha$ where each M_α is transitive, $M_\alpha \subseteq M_\beta$ for $\alpha \leq \beta$, and $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$ for limit α . One can discuss the question of “when do new Σ_1 truths about the reals (that is, about $\mathbb{R} \cup \{\mathbb{R}\}$) occur in M ?” Suppose, for example, that φ is a Σ_1 formula which has the set \mathbb{R} and an $x \in \mathbb{R}$ as parameters. If M_α is such that $M_\alpha \models \varphi$ and $M_\gamma \not\models \varphi$ for all $\gamma < \alpha$, then will say that M_α witnesses a new Σ_1 truth about the reals. Suppose that the ordinals $\alpha \leq \beta$ are such that M_α and $M_{\beta+1}$ both witness *new* Σ_1 truths about the reals. If, in addition, M_α and M_β both satisfy the *same* Σ_1 truths about the reals, then we will call the interval $[\alpha, \beta]$ a $\Sigma_1(M)$ -gap, where $[\alpha, \beta]$ is defined to be the set of ordinals $[\alpha, \beta] = \{\gamma \in \text{OR} : \alpha \leq \gamma \leq \beta\}$.

We will now review the concept of a Σ_1 -gap in $L(\mathbb{R})$. Then we will focus on the notion of a Σ_1 -gap in an iterable real premouse.

3.1. Σ_1 -gaps in $L(\mathbb{R})$. Steel [12] develops a fine structure theory for $L(\mathbb{R})$ and a Levy hierarchy for $L(\mathbb{R})$ and then solves the problem of finding scales of minimal complexity in $L(\mathbb{R})$. Given a set of reals $A \in L(\mathbb{R})$, using the reflection properties of the Levy hierarchy for $L(\mathbb{R})$, Steel identifies the first level $\Sigma_n(J_\alpha(\mathbb{R}))$ at which a scale on A is definable. This level occurs very close to the first ordinal α such that $A \in J_\alpha(\mathbb{R})$ and for some Σ_1 formula (allowing \mathbb{R} to appear as a constant) $\varphi(v)$ and for some $x \in \mathbb{R}$, one has $J_{\alpha+1}(\mathbb{R}) \models \varphi[x]$ and yet $J_\alpha(\mathbb{R}) \not\models \varphi[x]$. So in $L(\mathbb{R})$ there is a close connection between obtaining scales of minimal complexity and new Σ_1 truths about the reals. Accordingly, Steel introduces the following definition (see [12, Definition 2.2]).

Definition 3.1. Let $\alpha \leq \beta$ be ordinals. The interval $[\alpha, \beta]$ is a $\Sigma_1(L(\mathbb{R}))$ -gap if and only if

- (1) $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$
- (2) $J_{\alpha'}(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_\alpha(\mathbb{R})$ for all $\alpha' < \alpha$
- (3) $J_\beta(\mathbb{R}) \not\prec_1^{\mathbb{R}} J_{\beta'}(\mathbb{R})$ for all $\beta' > \beta$.

Remark 3.2. As noted earlier, we always allow the parameter \mathbb{R} to appear as a constant in our relevant languages. Thus, the statement $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$ is in fact equivalent to the statement $J_\alpha(\mathbb{R}) \prec_1^{\mathbb{R} \cup \{\mathbb{R}\}} J_\beta(\mathbb{R})$.

Let $\delta \in \text{OR}$ be the least such that $J_\delta(\mathbb{R}) \prec_1^{\mathbb{R}} L(\mathbb{R})$. One can show that the $\Sigma_1(L(\mathbb{R}))$ -gaps partition δ . Moreover, since $J_\delta(\mathbb{R}) \prec_1^{\mathbb{R}} J_\beta(\mathbb{R})$ for all ordinals $\beta > \delta$, it follows that δ starts a $\Sigma_1(L(\mathbb{R}))$ -gap which has “no end”. In Section 4, however, we will show that this particular gap will have a “proper” ending in any iterable real premouse.

3.2. Σ_1 -gaps in $K(\mathbb{R})$. Since $K(\mathbb{R})$ is the union of real 1-mice, the notion of a Σ_1 -gap in $K(\mathbb{R})$ reduces to discussing such gaps in iterable real premice. We recall that an iterable real premouse $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is a structure, consisting of sets constructible above the reals relative to the measure μ , with a Jensen hierarchy similar to that in subsection 2.1 (see Section 1 of [3]). We denote the α^{th} level of this hierarchy by \mathcal{N}^α . This avoids any confusion with the notation \mathcal{N}_α which denotes the α^{th} premouse iterate of \mathcal{N} . In Section 4 we shall identify the pointclasses of the form $\Sigma_n(\mathcal{N}^\alpha)$ which have the scale property.

Σ_1 -gaps in iterable real premice. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and let \mathcal{M} be the iterable real premouse defined by $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu) = \mathcal{N}^{\kappa+1}$. It can be shown that every set of reals in $L(\mathbb{R})$ is in M and that for all $\alpha \leq \kappa$, $J_\alpha(\mathbb{R}) = M^\alpha$. In fact, letting δ be as in the above subsection 3.1, one can show that $\delta < \kappa$ and $J_\delta(\mathbb{R}) \not\prec_1^{\mathbb{R}} \mathcal{M}$. Thus, in \mathcal{M} the ordinal κ is the end of the Σ_1 -gap that began with δ and the ordinal $\kappa + 1$ starts a new Σ_1 -gap. Furthermore, one can show that there is a set of reals $A \notin L(\mathbb{R})$ such that A has a $\Sigma_1(\mathcal{M})$ scale.² Consequently, the connection between new scales and Σ_1 truths continues.

²Namely, $A = \mathbb{R}^\#$. The iterable real premouse \mathcal{M} is “ \mathbb{R} -sharplike” (see footnote 9).

We will show that in iterable real premouse there is a close connection between obtaining scales of minimal complexity and new Σ_1 truths about the reals. The following definition is a straightforward generalization of Definition 3.1.

Definition 3.3. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse. Let $\kappa < \alpha \leq \beta \leq \widehat{\text{OR}}^{\mathcal{N}}$. We shall say that the interval $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap if and only if

- (1) $\mathcal{N}^\alpha \prec_1^{\mathbb{R}} \mathcal{N}^\beta$
- (2) $\mathcal{N}^{\alpha'} \not\prec_1^{\mathbb{R}} \mathcal{N}^\alpha$ for all $\alpha' < \alpha$
- (3) $\mathcal{N}^\beta \not\prec_1^{\mathbb{R}} \mathcal{N}^{\beta'}$ for all $\beta' > \beta$ where $\beta' \leq \widehat{\text{OR}}^{\mathcal{N}}$.

When $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap we shall say that α *begins* the gap and that β *ends* the gap. In addition, if $\alpha < \widehat{\text{OR}}^{\mathcal{N}}$ then we shall say that α *properly begins* the gap and, when the context is clear, we will say that α is proper. When $\beta < \widehat{\text{OR}}^{\mathcal{N}}$ we will say that β *properly ends* the gap and, when the context is clear, we shall say that β is proper.

Remark 3.4. Two observations concerning Definition 3.3:

- (1) We allow for the possibility that $[\alpha, \alpha]$ is a $\Sigma_1(\mathcal{N})$ -gap when $\alpha \leq \widehat{\text{OR}}^{\mathcal{N}}$.
- (2) The ordinal $\delta = \widehat{\text{OR}}^{\mathcal{N}}$ always ends a $\Sigma_1(\mathcal{N})$ -gap. More specifically, either (i) $[\delta, \delta]$ is a $\Sigma_1(\mathcal{N})$ -gap, or (ii) $[\alpha, \delta]$ is a $\Sigma_1(\mathcal{N})$ -gap for some $\alpha < \delta$.

Remark 3.5. Suppose that an iterable real premouse \mathcal{M} is a proper extension of \mathcal{N} . If α begins a $\Sigma_1(\mathcal{N})$ -gap then α will also begin a $\Sigma_1(\mathcal{M})$ -gap. Also, if $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap and $\beta < \widehat{\text{OR}}^{\mathcal{N}}$, then $[\alpha, \beta]$ will likewise be a $\Sigma_1(\mathcal{M})$ -gap. If $\beta = \widehat{\text{OR}}^{\mathcal{N}}$, then it is possible for the interval $[\alpha, \beta]$ to fail to be a $\Sigma_1(\mathcal{M})$ -gap, because the end of the corresponding $\Sigma_1(\mathcal{M})$ -gap may be greater than β .

Definition 3.6. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse. For $\gamma < \widehat{\text{OR}}^{\mathcal{N}}$ we shall say that $\underline{\Sigma}_n(\mathcal{N}^\gamma)$ and $\underline{\Pi}_n(\mathcal{N}^\gamma)$, for $n \geq 1$, are *internal* pointclasses. On the other hand, we shall call $\underline{\Sigma}_n(\mathcal{N})$ and $\underline{\Pi}_n(\mathcal{N})$, for $n \geq 1$, *external* pointclasses.

4. COMPLEXITY OF SCALES IN AN ITERABLE REAL PREMUSE

In this section we shall assume that $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is an iterable real premouse. We shall use the notions developed in the previous sections to identify those levels of the Levy hierarchy for \mathcal{N} which have the scale property.

4.1. Scales at the beginning of a gap. Recall that an iterable real premouse \mathcal{M} is a 1-mouse if $\omega\rho_{\mathcal{M}} \leq \kappa^{\mathcal{M}}$ (see Definition 1.13). Definition 3.9 of [1] describes the Σ_n hull of a premouse. Let \mathcal{M} be a 1-mouse, let $\mathcal{H} = \text{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \omega\rho_{\mathcal{M}} \cup \{p_{\mathcal{M}}\})$, and let \mathfrak{C} be the transitive collapse of \mathcal{H} . It follows that \mathfrak{C} is a real premouse, denoted by $\mathfrak{C}(\mathcal{M})$.

Theorem 4.1. Suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. Let $\mathcal{C} = (C, \mathbb{R}, \kappa^{\mathcal{C}}, \mu^{\mathcal{C}})$ be the transitive collapse of $\text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R})$. Then

- (i) \mathcal{C} is an iterable real premouse and $\mathcal{C}_\xi = \mathcal{N}^\alpha$ for some ordinal ξ .
- (ii) There exists a set of reals A such that $A \in \Sigma_1(\mathcal{N}^\alpha)$ and $A \notin \mathcal{N}^\alpha$.
- (iii) \mathcal{N}^α is a real 1-mouse, $\mathfrak{C}(\mathcal{N}^\alpha) = \mathcal{C}$, $\rho_{\mathcal{N}^\alpha} = 1$ and $p_{\mathcal{N}^\alpha} = \emptyset$.
- (iv) $\underline{\Sigma}_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$, as pointclasses.
- (v) $\mathcal{P}(\mathbb{R}) \cap \mathcal{N}^\alpha \subset \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$.

Proof. Let α , \mathcal{N} and \mathcal{C} be as stated in the theorem. We prove items (i)–(iv).

(i) Let $\sigma: \mathcal{C} \xrightarrow[\Sigma_1]{\mathcal{N}^\alpha} \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R})$ be the inverse of the collapse map $\pi: \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R}) \rightarrow \mathcal{C}$. It follows that \mathcal{C} is a real premouse. Because σ is a Σ_1 embedding, we have that the induced map $\sigma: F^{\mathcal{C}} \rightarrow F^{\mathcal{N}^\alpha}$ is \leq -extendible (see Definition 3.48 of [1]). Thus, Theorem 3.49 of [1] implies that \mathcal{C} is an iterable real premouse. Note that $\mathcal{C} = \text{Hull}_1^{\mathcal{C}}(\mathbb{R})$. We now conclude from Theorem 2.32 of [3] that there is an ordinal ξ such that the ξ^{th} premouse iterate \mathcal{C}_ξ is an initial segment of \mathcal{N}^α .

Because $\pi_{0\xi}: \mathcal{C} \xrightarrow{\Sigma_1} \mathcal{C}_\xi$, where $\pi_{0\xi}$ is the premouse embedding of \mathcal{C} into \mathcal{C}_ξ , it follows that $\mathcal{C}_\xi \prec_1^{\mathbb{R}} \mathcal{N}^\alpha$ and, since α begins a $\Sigma_1(\mathcal{N})$ -gap, we see that $\mathcal{C}_\xi = \mathcal{N}^\alpha$.

(ii) We first define a set of reals A such that $A \in \Sigma_1(\mathcal{C})$ and $A \notin C$. Let $g = h_C \upharpoonright (\mathbb{R} \times \mathbb{R})$ be the partial map obtained by restricting the canonical $\Sigma_1(\mathcal{C})$ Skolem function h_C (in [1] see Definition 3.8 and Lemma 3.10). Because $\mathcal{C} = \text{Hull}_1^{\mathcal{C}}(\mathbb{R})$, it follows that $g: \mathbb{R} \times \mathbb{R} \xrightarrow{\text{onto}} C$. Define $A \subseteq \mathbb{R}$ by

$$x \in A \text{ iff } x \notin g(x_0, x_1).$$

Note that $A \in \Sigma_1(\mathcal{C})$ and, since g is onto, $A \notin C$. Because $A \in \Sigma_1(\mathcal{C})$, we have that $A \in \Sigma_1(\mathcal{N}^\alpha)$. Also, since $\mathcal{C}_\xi = \mathcal{N}^\alpha$, Lemma 2.11(4) of [3] implies that $A \notin \mathcal{N}^\alpha$.

(iii) It follows from (i) and (ii) that \mathcal{N}^α is a real 1-mouse, $\mathcal{C} = \mathfrak{C}(\mathcal{N}^\alpha)$, $\rho_{\mathcal{N}^\alpha} = 1$ and $p_{\mathcal{N}^\alpha} = \emptyset$.

(iv) Because $\sigma: \mathcal{C} \xrightarrow{\Sigma_1} \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R})$, we see that $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}) = \Sigma_1(\mathcal{C}, \mathbb{R})$. In addition, since $\mathcal{C} = \text{Hull}_1^{\mathcal{C}}(\mathbb{R})$, we have that $\Sigma_1(\mathcal{C}) = \Sigma_1(\mathcal{C}, \mathbb{R})$. Since $\mathcal{C}_\xi = \mathcal{N}^\alpha$, Corollary 2.14 of [3] implies that $\Sigma_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{C})$. Therefore, $\Sigma_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$.

(v) This follows immediately from (ii) and (iv). \square

Theorem 4.2. *Suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. If $\mathcal{N}^\alpha \models \text{AD}$, then $\Sigma_1(\mathcal{N}^\alpha)$ has the scale property.*

Proof. Corollary 1.2 states that $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$ has the scale property. Therefore, $\Sigma_1(\mathcal{N}^\alpha)$ has the scale property by (iv) of Theorem 4.1. \square

When α begins a $\Sigma_1(\mathcal{N})$ -gap, we conclude that $\Pi_1(\mathcal{N}^\alpha)$ does not have the scale property (see [11, 4B.13]). The next classes to consider are the pointclasses $\Sigma_n(\mathcal{N}^\alpha)$ where $n > 1$ and α properly begins a $\Sigma_1(\mathcal{N})$ -gap.

Definition 4.3. Let $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ be a pointclass. Define

$$\mathfrak{B}(\Gamma) = \{A \subseteq \mathbb{R} : A \text{ is a boolean combination of } \Gamma \text{ sets}\}.$$

$\mathfrak{B}(\Gamma)$ is a pointclass that is closed under the operations of intersection, union and complement.

Definition 4.4. Given a pointclass Γ , define $\Sigma_n^*(\Gamma)$ by the following induction on $n \in \omega$,

$$\begin{aligned} \Sigma_0^*(\Gamma) &= \Gamma \\ \Pi_n^*(\Gamma) &= \neg \Sigma_n^*(\Gamma) \\ \Sigma_{n+1}^*(\Gamma) &= \exists^{\mathbb{R}} \Pi_n^*(\Gamma) \end{aligned}$$

Remark 4.5. Let $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ be a real premouse. Applying Definition 4.4 to the pointclass $\Gamma = \Sigma_1(\mathcal{M}, \mathbb{R})$ we obtain the following equations:

$$\begin{aligned} \Sigma_0^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \Sigma_1(\mathcal{M}, \mathbb{R}) \\ \Pi_0^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \Pi_1(\mathcal{M}, \mathbb{R}) \\ \Sigma_1^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \exists^{\mathbb{R}} \Pi_1(\mathcal{M}, \mathbb{R}) \\ \Pi_1^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \forall^{\mathbb{R}} \Sigma_1(\mathcal{M}, \mathbb{R}) \\ \Sigma_2^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \exists^{\mathbb{R}} \forall^{\mathbb{R}} \Sigma_1(\mathcal{M}, \mathbb{R}) \\ \Pi_2^*(\Sigma_1(\mathcal{M}, \mathbb{R})) &= \forall^{\mathbb{R}} \exists^{\mathbb{R}} \Pi_1(\mathcal{M}, \mathbb{R}). \end{aligned}$$

Consequently, assuming enough determinacy, Corollary 1.2 and the Second Periodicity Theorem of Moschovakis [11, Theorems 6C.2 and 6C.3] imply that the pointclasses

$$\Sigma_0^*(\Sigma_1(\mathcal{M}, \mathbb{R})), \Pi_1^*(\Sigma_1(\mathcal{M}, \mathbb{R})), \Sigma_2^*(\Sigma_1(\mathcal{M}, \mathbb{R}))$$

have the scale property.

We shall motivate our next proposition with two examples. Let $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ be a real premouse. Suppose that $\mathcal{M} = \text{Hull}_1^{\mathcal{M}}(\mathbb{R})$ and let $g: \mathbb{R} \rightarrow M$ be the *partial* function defined by $g(x) = h_{\mathcal{M}}(x_0, x_1)$. Thus, g is onto M with $\Sigma_1(\mathcal{M})$ graph. Let $D = \text{dom}(g)$ and note that $D \in \Sigma_1(\mathcal{M})$.

Example 1. Now consider the pointclass $\Sigma_2(\mathcal{M})$. Let $A \in \Sigma_2(\mathcal{M})$. Then there is a Π_1 formula $\varphi(u, v, w)$ and a $k \in M$ such that

$$x \in A \text{ iff } \mathcal{M} \models (\exists u)\varphi(u, x, k)$$

for all $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ be such that $g(y) = k$, where g is defined above. Since

$$\begin{aligned} x \in A &\text{ iff } \mathcal{M} \models (\exists u)\varphi(u, x, g(y)) \\ &\text{ iff } \mathcal{M} \models (\exists z \in \mathbb{R})[z \in D \wedge (\forall a)(\forall b)((a = g(z) \wedge b = g(y)) \rightarrow \varphi(a, x, b))], \end{aligned}$$

we conclude that $A \in \exists^{\mathbb{R}}(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$. By definition,

$$\exists^{\mathbb{R}}(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R}))) = \Sigma_1^*(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$$

and so, $\Sigma_2(\mathcal{M}) \subseteq \Sigma_1^*(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$.

Example 2. In addition, consider the pointclass $\Sigma_3(\mathcal{M})$. Let $A \in \Sigma_3(\mathcal{M})$. Then there is a Σ_1 formula $\psi(t, u, v, w)$ and a $k \in M$ such that

$$x \in A \text{ iff } \mathcal{M} \models (\exists t)(\forall u)\psi(t, u, x, k)$$

for all $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ be such that $g(y) = k$. Since

$$\begin{aligned} x \in A &\text{ iff } \mathcal{M} \models (\exists t)(\forall u)\psi(t, u, x, g(y)) \\ &\text{ iff } \mathcal{M} \models (\exists p \in \mathbb{R})(\forall z \in \mathbb{R})[p \in D \wedge (z \in D \rightarrow \\ &\quad (\exists a)(\exists b)(\exists c)(a = g(p) \wedge b = g(z) \wedge c = g(y) \wedge \psi(a, b, x, c)))] \end{aligned}$$

we conclude that $A \in \exists^{\mathbb{R}}\forall^{\mathbb{R}}(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$. By definition,

$$\exists^{\mathbb{R}}\forall^{\mathbb{R}}(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R}))) = \Sigma_2^*(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$$

and so, $\Sigma_3(\mathcal{M}) \subseteq \Sigma_2^*(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$.

This completes our two examples. The following proposition should now be clear.

Proposition 4.6. Let $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$ be a real premouse. Assume that $\mathcal{M} = \text{Hull}_1^{\mathcal{M}}(\mathbb{R})$. Then $\Sigma_{n+1}(\mathcal{M}) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})))$ for all $n \geq 1$.

Lemma 4.7. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. Then

$$\Sigma_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})))$$

for all $n \geq 1$.

Proof. Since α begins a $\Sigma_1(\mathcal{N})$ -gap. Let $\mathcal{C} = (C, \mathbb{R}, \kappa^{\mathcal{C}}, \mu^{\mathcal{C}})$ be the transitive collapse of $\mathcal{H} = \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R})$. By Theorem 4.1, there is a premouse iterate \mathcal{C}_ξ such that $\mathcal{C}_\xi = \mathcal{N}^\alpha$. Let $n \geq 1$.

Claim. $\Sigma_1(\mathcal{C}, \mathbb{R}) = \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$ and $\Sigma_{n+1}(\mathcal{C}) = \Sigma_{n+1}(\mathcal{N}^\alpha)$ as pointclasses.

Proof of Claim. Clearly, $\Sigma_1(\mathcal{C}, \mathbb{R}) = \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$. Corollary 2.20 of [3] implies that $\Sigma_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_{n+1}(\mathcal{C})$. Let $\mathcal{H}^\alpha = \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R})$. Note that \mathcal{C} and \mathcal{H}^α are isomorphic structures and thus, $\Sigma_{n+1}(\mathcal{C}) = \Sigma_{n+1}(\mathcal{H}^\alpha)$. Because \mathcal{H}^α is Σ_1 definable over \mathcal{N}^α , we have that $\Sigma_{n+1}(\mathcal{H}^\alpha) \subseteq \Sigma_{n+1}(\mathcal{N}^\alpha)$. Therefore, $\Sigma_{n+1}(\mathcal{C}) = \Sigma_{n+1}(\mathcal{N}^\alpha)$. \square

Proposition 4.6 asserts that $\Sigma_{n+1}(\mathcal{C}) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_1(\mathcal{C}, \mathbb{R})))$. Thus, the Claim implies that $\Sigma_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})))$. \square

Definition 4.8. Let \mathcal{M} be a real premouse. We say that \mathcal{M} is \mathbb{R} -collectible if for every Σ_0 formula $\varphi(x, v)$ (allowing arbitrary parameters in M) we have that

$$\mathcal{M} \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v) \text{ implies } \mathcal{M} \models (\exists w)(\forall x \in \mathbb{R})(\exists v \in w)\varphi(x, v).$$

Lemma 4.9. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. Then \mathcal{N}^α is \mathbb{R} -collectible if and only if $\mathfrak{C}(\mathcal{N}^\alpha)$ is \mathbb{R} -collectible.

Proof. Suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. Let $\mathcal{C} = (C, \mathbb{R}, \kappa^{\mathcal{C}}, \mu^{\mathcal{C}})$ be the transitive collapse of $\mathcal{H} = \text{Hull}_1^{\mathcal{N}^\alpha}(\mathbb{R}) = (H, \mathbb{R}, \kappa^{\mathcal{H}}, \mu^{\mathcal{H}})$. By Theorem 4.1, $\mathcal{C} = \mathfrak{C}(\mathcal{N}^\alpha)$ and $\mathcal{C}_\xi = \mathcal{N}^\alpha$ for some ordinal ξ .

(\Rightarrow). Assume \mathcal{N}^α is \mathbb{R} -collectible. We will show that \mathcal{H} is \mathbb{R} -collectible. It will then follow that $\mathfrak{C}(\mathcal{N}^\alpha)$ is \mathbb{R} -collectible. Note that $\mathcal{H} \prec_1 \mathcal{N}^\alpha$ and $\mathbb{R} \cup \{\mathbb{R}\} \in H$. For simplicity let $\varphi(x, v_1, v_2)$ be Σ_0 and let $a \in H$. Suppose that

$$(4.1) \quad \mathcal{H} \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v, a).$$

Since $\mathcal{H} \prec_1 \mathcal{N}^\alpha$, (4.1) implies that

$$\mathcal{N}^\alpha \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v, a).$$

Then, because \mathcal{N}^α is \mathbb{R} -collectible,

$$\mathcal{N}^\alpha \models (\exists w)(\forall x \in \mathbb{R})(\exists v \in w)\varphi(x, v, a).$$

Since $\mathcal{H} \prec_1 \mathcal{N}^\alpha$, it follows that

$$\mathcal{H} \models (\exists w)(\forall x \in \mathbb{R})(\exists v \in w)\varphi(x, v, a).$$

Therefore, $\mathfrak{C}(\mathcal{N}^\alpha)$ is \mathbb{R} -collectible.

(\Leftarrow). Assume $\mathcal{C} = \mathfrak{C}(\mathcal{N}^\alpha)$ is \mathbb{R} -collectible. Let

$$\langle \langle \mathcal{C}_\gamma \rangle_{\gamma \in \text{OR}}, \langle \pi_{\gamma\beta} : \mathcal{C}_\gamma \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{C}_\beta \rangle_{\gamma \leq \beta \in \text{OR}} \rangle$$

be the premouse iteration of \mathcal{C} . Note that $\mathbb{R} \cup \{\mathbb{R}\} \in C_\gamma$ for all $\gamma \in \text{OR}$ (see [3, Lemma 2.11(3)]). We will show by induction on λ , that \mathcal{C}_λ is \mathbb{R} -collectible. We can then conclude that \mathcal{N}^α is \mathbb{R} -collectible, since $\mathcal{C}_\xi = \mathcal{N}^\alpha$. Now, for $\lambda = 0$, $\mathcal{C}_0 = \mathcal{C}$. Hence, \mathcal{C}_0 is \mathbb{R} -collectible by assumption.

SUCCESSOR CASE: Let $\lambda \geq 0$ and assume that \mathcal{C}_λ is \mathbb{R} -collectible. We will show that $\mathcal{C}_{\lambda+1}$ is \mathbb{R} -collectible. Let $\pi = \pi_{\lambda, \lambda+1}$. Thus, $\pi : \mathcal{C}_\lambda \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{C}_{\lambda+1}$. For simplicity let $\varphi(x, v_1, v_2)$ be Σ_0 and let $a \in C_{\lambda+1}$. Now suppose that

$$(4.2) \quad \mathcal{C}_{\lambda+1} \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v, a).$$

Since π is cofinal, (4.2) implies

$$(\forall x \in \mathbb{R})(\exists b \in C_\lambda) [\mathcal{C}_{\lambda+1} \models (\exists v \in \pi(b))\varphi(x, v, a)].$$

By Lemma 2.8(3) of [3], there is a function $f \in C_\lambda$ such that $\pi(f)(\kappa_\lambda) = a$. Thus by Theorem 2.4 of [3], we conclude that

$$(\forall x \in \mathbb{R})(\exists b_1, b_2 \in C_\lambda) [\mathcal{C}_\lambda \models (b_0 = \{\eta \in \underline{\kappa} : (\exists v \in b_1)\varphi(x, v, f(\eta))\} \wedge b_0 \in \mu)].$$

By our induction hypothesis, there exists a $w \in C_\lambda$ such that

$$\mathcal{C}_\lambda \models (\forall x \in \mathbb{R})(\exists b_1, b_2 \in w) [b_0 = \{\eta \in \underline{\kappa} : (\exists v \in b_1)\varphi(x, v, f(\eta))\} \wedge b_0 \in \mu].$$

It follows that

$$\mathcal{C}_{\lambda+1} \models (\forall x \in \mathbb{R})(\exists v \in \pi(w))\varphi(x, v, a).$$

This argument shows that $\mathcal{C}_{\lambda+1}$ is \mathbb{R} -collectible.

LIMIT CASE: Let $\lambda \geq 0$ be a limit ordinal and assume that \mathcal{C}_ν is \mathbb{R} -collectible for all $\nu < \lambda$. We will show that \mathcal{C}_λ is \mathbb{R} -collectible. For simplicity let $\varphi(x, v_1, v_2)$ be Σ_0 and let $a \in C_\lambda$. Suppose that

$$(4.3) \quad \mathcal{C}_\lambda \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v, a).$$

Because \mathcal{C}_λ is a direct limit, there is an ordinal $\nu < \lambda$ and an element $\hat{a} \in C_\nu$ such that $\pi_{\nu\lambda}(\hat{a}) = a$. Since $\pi_{\nu\lambda} : \mathcal{C}_\nu \xrightarrow[\Sigma_1]{} \mathcal{C}_\lambda$, we see from (4.3) that

$$\mathcal{C}_\nu \models (\forall x \in \mathbb{R})(\exists v)\varphi(x, v, \hat{a}).$$

By our induction hypothesis, there exists a $w \in C_\nu$ such that

$$\mathcal{C}_\nu \models (\forall x \in \mathbb{R})(\exists v \in w)\varphi(x, v, \hat{a}).$$

Therefore,

$$\mathcal{C}_\lambda \models (\forall x \in \mathbb{R})(\exists v \in \pi_{\nu\lambda}(w))\varphi(x, v, a).$$

This argument shows that \mathcal{C}_λ is \mathbb{R} -collectible. \square

Corollary 4.10. *Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. If \mathcal{N}^α is not \mathbb{R} -collectible, then there is a total function $k: \mathbb{R} \xrightarrow{\text{cofinal}} \text{OR}^{\mathcal{N}^\alpha}$ whose graph is $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$.*

Proof. Assume that \mathcal{N}^α is not \mathbb{R} -collectible. Let $\mathcal{C} = \mathfrak{C}(\mathcal{N}^\alpha)$ and let ξ be such that $\mathcal{C}_\xi = \mathcal{N}^\alpha$ (see (i) of Theorem 4.1). By Lemma 4.9, \mathcal{C} is not \mathbb{R} -collectible. It follows that there is a total function $k': \mathbb{R} \xrightarrow{\text{cofinal}} \text{OR}^{\mathcal{C}}$ whose graph is $\Sigma_1(\mathcal{C})$. Since there is a partial $\Sigma_1(\mathcal{C})$ map of \mathbb{R} onto \mathcal{C} , we see that the graph of k' is $\Sigma_1(\mathcal{C}, \mathbb{R})$. Because $\pi_{0\xi}: \mathcal{C} \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{N}^\alpha$, where $\pi_{0\xi}$ is the premouse embedding of \mathcal{C} into $\mathcal{C}_\xi = \mathcal{N}^\alpha$, we conclude that there is a total function $k: \mathbb{R} \xrightarrow{\text{cofinal}} \text{OR}^{\mathcal{N}^\alpha}$ whose graph is $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$ (k is just the interpretation of k' in \mathcal{N}^α). \square

Definition 4.11. Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. Then we shall say that

- α is *collectible* if and only if \mathcal{N}^α is \mathbb{R} -collectible,
- α is *uncollectible* if and only if \mathcal{N}^α is not \mathbb{R} -collectible.

Remark. Suppose that \mathcal{N} is an iterable real premouse and $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap. If $\alpha < \beta$ then α is collectible; however, if $\alpha = \beta$ then α can be uncollectible.

Lemma 4.12. *Suppose that α begins a $\Sigma_1(\mathcal{N})$ -gap. If α is uncollectible, then*

- (a) $\Sigma_{n+1}(\mathcal{N}^\alpha) = \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}))$
- (b) $\Pi_{n+1}(\mathcal{N}^\alpha) = \Pi_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}))$

(as pointclasses) for all $n \geq 0$.

Proof. By (iv) of Theorem 4.1, $\Sigma_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$. Thus, the Lemma holds for $n = 0$, by Definition 4.4. So we assume $n \geq 1$.

It should be clear that

$$\begin{aligned} \Sigma_{n+1}(\mathcal{N}^\alpha) &\supseteq \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})) \\ \Pi_{n+1}(\mathcal{N}^\alpha) &\supseteq \Pi_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})) \end{aligned}$$

and thus we shall show that

$$\begin{aligned} \Sigma_{n+1}(\mathcal{N}^\alpha) &\subseteq \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})) \\ \Pi_{n+1}(\mathcal{N}^\alpha) &\subseteq \Pi_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})). \end{aligned}$$

In fact, it is sufficient to prove that $\Sigma_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}))$. Now, to simplify the notation slightly, for any real premouse \mathcal{M} let

$$\mathfrak{B}_1(\mathcal{M}) = \mathfrak{B}(\Sigma_1(\mathcal{M}, \mathbb{R})).$$

By Lemma 4.7, $\Sigma_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_n^*(\mathfrak{B}_1(\mathcal{N}^\alpha))$.

Claim. $\mathfrak{B}_1(\mathcal{N}^\alpha) \subseteq \forall^{\mathbb{R}} \Sigma_1(\mathcal{N}^\alpha, \mathbb{R}) \cap \exists^{\mathbb{R}} \Pi_1(\mathcal{N}^\alpha, \mathbb{R})$.

Proof of Claim. Because $\mathfrak{B}_1(\mathcal{N}^\alpha)$ is closed under complementation, it is sufficient to show that

$$(*) \quad \mathfrak{B}_1(\mathcal{N}^\alpha) \subseteq \exists^{\mathbb{R}} \Pi_1(\mathcal{N}^\alpha, \mathbb{R}).$$

To show (*), it is enough to establish that $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}) \subseteq \exists^{\mathbb{R}} \Pi_1(\mathcal{N}^\alpha, \mathbb{R})$. Let $Q \in \Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$, say

$$y \in Q \text{ iff } \mathcal{N}^\alpha \models \varphi(y, z)$$

for all $y \in \mathbb{R}$, where $\varphi \in \Sigma_1$ and $z \in \mathbb{R}$. We will show that $Q \in \exists^{\mathbb{R}} \Pi_1(\mathcal{N}^\alpha, \mathbb{R})$. Corollary 4.10 implies that there is a total function $k: \mathbb{R} \xrightarrow{\text{cofinal}} \text{OR}^{\mathcal{N}^\alpha}$ whose graph is $\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})$ and such that $k(x) > \kappa$ for all $x \in \mathbb{R}$.

Let $\langle S_\gamma : \gamma < \text{OR}^{\mathcal{N}^\alpha} \rangle$ be the $\Sigma_1(\mathcal{N}^\alpha)$ increasing sequence of transitive sets (see Section 1 of [3]) such that

- $\mathbb{R} \in S_\gamma \in N^\alpha$, for all $\eta \in \text{OR}^{\mathcal{N}^\alpha}$
- $N^\alpha = \bigcup_{\eta \in \zeta} S_\eta$, where $\zeta = \text{OR}^{\mathcal{N}^\alpha}$.

For all ordinals η such that $\kappa \in \eta \in \text{OR}^{\mathcal{N}^\alpha}$ we let $\mathbb{S}_\eta = (S_\eta, \mathbb{R}, \kappa, \mu \cap S_\eta)$. Note that $\mathbb{S}_\eta \in N^\alpha$ and the sequence $\langle \mathbb{S}_\eta : \kappa < \eta < \text{OR}^{\mathcal{N}^\alpha} \rangle$ is $\Sigma_1(\mathcal{N}^\alpha)$. Now, for $y \in \mathbb{R}$

$$\begin{aligned} y \in Q &\text{ iff } \mathcal{N}^\alpha \models \varphi(y, z) \\ &\text{ iff } (\exists x \in \mathbb{R}) [\mathbb{S}_{k(x)} \models \varphi(y, z)] \\ &\text{ iff } \mathcal{N}^\alpha \models (\exists x \in \mathbb{R})(\forall \mathcal{S})(\forall \eta) [(\eta = k(x) \wedge \mathcal{S} = \mathbb{S}_\eta) \rightarrow \varphi(y, z)^{\mathcal{S}}]^3 \end{aligned}$$

where $\varphi(y, z)^{\mathcal{S}}$ is the “relativization” of φ to \mathcal{S} . Therefore, $Q \in \exists^{\mathbb{R}}\Pi_1(\mathcal{N}^\alpha, \mathbb{R})$ and this completes the proof of the Claim. \square

We conclude from the Claim that

$$\Sigma_n^*(\mathfrak{B}_1(\mathcal{N}^\alpha)) \subseteq \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R}))$$

and thus,

$$\underline{\Sigma}_{n+1}(\mathcal{N}^\alpha) \subseteq \Sigma_n^*(\Sigma_1(\mathcal{N}^\alpha, \mathbb{R})).$$

This completes the proof of the Lemma. \square

Theorem 4.13. *Suppose that α properly begins a $\Sigma_1(\mathcal{N})$ -gap, α is uncollectible, and $\mathcal{N}^{\alpha+1} \models \text{AD}$. Then the pointclasses*

$$\underline{\Sigma}_{2n+1}(\mathcal{N}^\alpha) \text{ and } \underline{\Pi}_{2n+2}(\mathcal{N}^\alpha)$$

have the scale property, for all $n \geq 0$.

Proof. The theorem follows from Lemma 4.12, Corollary 1.2 and the Second Periodicity Theorem of Moschovakis [11, Theorems 6C.2 and 6C.3] (see Remark 4.5). \square

When α properly begins a $\Sigma_1(\mathcal{N})$ -gap and is collectible, then Martin’s arguments in [10] give the following analogues of Theorem 2.7 and Corollary 2.8 of [12].

Theorem 4.14. *Suppose that α properly begins a $\Sigma_1(\mathcal{N})$ -gap, α is collectible, and $\mathcal{N}^{\alpha+1} \models \text{AD}$. Then there is a $\Pi_1(\mathcal{N}^\alpha)$ subset of $\mathbb{R} \times \mathbb{R}$ with no uniformization in $\mathcal{N}^{\alpha+1}$.*

Corollary 4.15. *Suppose that α properly begins a $\Sigma_1(\mathcal{N})$ -gap, α is collectible, and $\mathcal{N}^{\alpha+1} \models \text{AD}$. Then the pointclasses*

$$\underline{\Sigma}_{n+1}(\mathcal{N}^\alpha) \text{ and } \underline{\Pi}_n(\mathcal{N}^\alpha)$$

do not have the scale property for all $n \geq 1$.

4.2. Scales inside a gap. In this subsection we shall focus our attention on $\Sigma_1(\mathcal{N})$ -gaps $[\alpha, \beta]$ where $\alpha < \beta$. Thus, α is proper and collectible. Our first theorem extends the above Theorem 4.14. The proofs of Theorem 2.9 and Corollary 2.10 of [12] easily generalize to give the next two results.

Theorem 4.16. *Suppose that the $\Sigma_1(\mathcal{N})$ -gap $[\alpha, \beta]$ is such that $\alpha < \beta$ and $\mathcal{N}^{\alpha+1} \models \text{AD}$. Then there is a $\Pi_1(\mathcal{N}^\alpha)$ subset of $\mathbb{R} \times \mathbb{R}$ with no uniformization in $\underline{\Sigma}_1(\mathcal{N}^\beta)$.*

Corollary 4.17. *Suppose that the $\Sigma_1(\mathcal{N})$ -gap $[\alpha, \beta]$ is such that $\alpha < \beta$ and $\mathcal{N}^{\alpha+1} \models \text{AD}$. If $\alpha < \gamma < \beta$, then the pointclasses*

$$\underline{\Sigma}_n(\mathcal{N}^\gamma) \text{ and } \underline{\Pi}_n(\mathcal{N}^\gamma)$$

do not have the scale property for all $n \geq 1$.

Thus, no new scales exist properly inside a $\Sigma_1(\mathcal{N})$ -gap $[\alpha, \beta]$. Suppose β properly ends this $\Sigma_1(\mathcal{N})$ -gap. Do any of the pointclasses $\underline{\Sigma}_n(\mathcal{N}^\beta)$ or $\underline{\Pi}_n(\mathcal{N}^\beta)$ have the scale property? The results in [1] and [2] will be used to answer this question in our next subsection.

³If k were not total, then this condition would hold, vacuously, for all reals y .

4.3. Scales at the proper ending of a gap. Recall that $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is an iterable real premouse. In this subsection we shall again deal with $\Sigma_1(\mathcal{N})$ -gaps $[\alpha, \beta]$ where $\alpha < \beta$. The next theorem shows that the proper ending of a gap produces a real mouse.

Theorem 4.18. *Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap. Then \mathcal{N}^β is a real mouse.*

Proof. Recall that $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is an iterable real premouse. Since \mathcal{N}^β is an iterable real premouse, Theorem 4.1 of [1] asserts that \mathcal{N}^β is acceptable above the reals. We now show that \mathcal{N}^β is critical (see subsection 1.1). Because \mathcal{N}^β is a proper initial segment of \mathcal{N} , the statement “ \mathcal{N}^β is critical” is one whose truth can be locally verified in \mathcal{N} . Thus, for any premouse iterate \mathcal{N}_λ with corresponding premouse embedding $\pi: \mathcal{N} \xrightarrow[\Sigma_1]{} \mathcal{N}_\lambda$, we have that $\mathcal{N} \models \text{“}\mathcal{N}^\beta \text{ is critical”}$

if and only if $\mathcal{N}_\lambda \models \text{“}\pi(\mathcal{N}^\beta) \text{ is critical”}$. Consequently we can assume, without loss of generality, that κ is a regular cardinal⁴ greater than Θ , and that for all $X \subseteq \kappa$ in \mathcal{N} , we have $X \in \mu$ if and only if X contains a closed unbounded subset of κ .

We will now prove that \mathcal{N}^β is critical. Suppose, for a contradiction, that \mathcal{N}^β is not critical; that is, suppose that $\rho_{\mathcal{N}^\beta}^n > \kappa$ for all $n \in \omega$. For short, let $\mathcal{M} = \mathcal{N}^\beta$. Since $\mathcal{M} \in \mathcal{N}$ it follows for each n that $\mathcal{M}^n \in \mathcal{N}$ and thus, for each n the truth of “ $\rho_{\mathcal{N}^\beta}^n > \kappa$ ” can be verified in \mathcal{N} . Hence, for every n , we have that $\mathcal{N} \models \text{“}\rho_{\mathcal{N}^\beta}^n > \kappa$ ” and the statement “ $\rho_{\mathcal{N}^\beta}^n > \kappa$ ” is equivalent to a local Σ_1 condition over \mathcal{N} .

Because β properly ends the $\Sigma_1(\mathcal{N})$ -gap, there is a real x and a Σ_1 formula $\varphi(v)$ in the language $\mathcal{L} = \{\in, \mathbb{R}, \underline{\kappa}, \mu\}$ such that $\mathcal{N}^{\beta+1} \models \varphi(x)$ and $\mathcal{N}^\beta \not\models \varphi(x)$. Thus, there is a $k \in \omega$ such that $(S_{\beta+k}^{\mathcal{N}}(\mathbb{R}), \mathbb{R}, \kappa, \mu) \models \varphi(x)$ (see subsection 3.1 of [1]). By Proposition 3.85 of [1], there is a formula $\chi(v)$ in the language⁵ \mathcal{L}^μ such that $(\mathcal{N}^\beta, \mu^{\beta+1}) \models \chi(x)$ and $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \not\models \chi(x)$ for all $\gamma < \beta$. Because $\rho_{\mathcal{N}^\beta}^n > \kappa$ for all $n \in \omega$, we conclude that there is a Σ_ω formula⁶ $\psi(v)$ in the language \mathcal{L} such that $\mathcal{N}^\beta \models \psi(x)$ and for all $\gamma \leq \beta$, if $\mathcal{N}^\gamma \models \psi(x)$ then $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \models \chi(x)$.

Since $\rho_{\mathcal{N}^\beta}^n > \kappa$ for all $n \in \omega$, the proof of Theorem 3.89 of [1] applies and shows that \mathcal{N}^β is n -iterable for each n . Now, let $m \in \omega$ be so that $\chi(v)$ is Σ_m . Let $n \geq 1$ be fixed for a moment. Let $\mathcal{H} = \text{Hull}_1^{\mathcal{M}^n}(\mathbb{R})$. Hence, $\mathcal{H} \prec_1 \mathcal{M}^n$. Let \mathcal{C}^n be the transitive collapse of \mathcal{H} . By Lemma 3.64 of [1] there is a decoding \mathcal{C} of \mathcal{C}^n and a map $\sigma: \mathcal{C} \xrightarrow[\Sigma_{n+1}]{} \mathcal{M}$. Hence, \mathcal{C} is

an iterable premouse and is acceptable above the reals. It thus follows, from the definition of \mathcal{H} , that \mathcal{C} is critical and so, $n(\mathcal{C})$ is defined. Now, assume that n is large enough to ensure that $n = n(\mathcal{C}) > m$. This assumption implies that $\mathcal{C} \models \chi(x)$ and $\mathcal{C} \models \neg\varphi(x)$. Theorem 3.93 of [1] implies that \mathcal{C} is n -iterable. Therefore, \mathcal{C} is a real mouse.

Let \mathcal{C}_κ be the κ^{th} mouse iterate of \mathcal{C} . Since $\kappa > \Theta$, our assumption on κ implies that \mathcal{C}_κ and \mathcal{N} are comparable (see [3, Definition 2.23]).

Claim. $\mathcal{C}_\kappa = \mathcal{N}^\beta$.

Proof. Since $\mathcal{C} \models \neg\varphi(x)$, we see that $\mathcal{C}_\kappa \models \neg\varphi(x)$. If \mathcal{N}^β were a proper initial segment of \mathcal{C}_κ , then since $\mathcal{N}^\beta \models \varphi(x)$, we would conclude that $\mathcal{C}_\kappa \models \varphi(x)$, as $\varphi(v)$ is Σ_1 . But this is not possible, because $\mathcal{C}_\kappa \models \neg\varphi(x)$. Thus, we must have that $\mathcal{C}_\kappa = \mathcal{N}^\gamma$ for some $\gamma \leq \beta$. Since $\mathcal{C} \models \chi(x)$ and $\chi(v)$ is Σ_m , it follows that $\mathcal{C}_\kappa \models \chi(x)$ because $n(\mathcal{C}) > m$. Consequently, $\mathcal{N}^\gamma \models \chi(x)$ and, as noted above, we have that $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \models \psi(x)$. Hence, by Proposition 3.85 of [1] we have $\mathcal{N}^{\gamma+1} \models \varphi(x)$. Therefore, $\mathcal{N}^\gamma = \mathcal{N}^\beta$ and this completes the proof of the claim. \square

Since a mouse iterate of a mouse is again a mouse, we conclude from the Claim that \mathcal{N}^β must be critical. This contradiction forces us to conclude that if β properly ends a $\Sigma_1(\mathcal{N})$ -gap, then \mathcal{N}^β is critical. Theorem 3.89 of [1] now implies that \mathcal{N}^β is a real mouse. \square

Theorem 4.19. *Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap, where $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$, and let $n = n(\mathcal{N}^\beta)$. Then $(\exists A \subseteq \mathbb{R}) (A \notin N^{\beta+1} \setminus N^\beta)$ if and only if $\rho_{\mathcal{N}^\beta}^{n+1} < \kappa$.*

⁴This appeal to AC is removable. Let $M = L(\mathcal{N})$. One can prove the theorem in a ZFC-generic extension $M[G]$ and thus, by absoluteness, the result holds in $L(\mathcal{N})$ and hence in V .

⁵The language \mathcal{L}^μ has a quantifier symbol μ (see subsection 1.1).

⁶The formula $\psi(v)$ is constructed by induction on the complexity of $\chi(v)$. See [4, Cor. 1.33] and [4, Cor. 2.13].

Proof. Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap, where $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$. Theorem 4.18 implies that \mathcal{N}^β is a real mouse and so, $\mathfrak{n} = n(\mathcal{N}^\beta)$ is defined. Thus, $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} \leq \kappa$. If there exists a new set of reals in $\mathcal{N}^{\beta+1}$ which is not in \mathcal{N}^β , then Lemma 4.3 of [1] implies that $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$.

Now assume that $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$. We prove that there is a set of reals A in $\mathcal{N}^{\beta+1}$ which is not in \mathcal{N}^β . As in the proof of the previous theorem, we shall assume⁷ that κ is a regular cardinal greater than Θ and that for all $X \subseteq \kappa$ in \mathcal{N} we have $X \in \mu$ if and only if X contains a closed unbounded subset of κ . Let $\mathfrak{C} = \mathfrak{C}(\mathcal{N}^\beta)$ be the core of the real mouse \mathcal{N}^β (see subsection 3.4.2 of [1]), let $\kappa_0 = \kappa^\mathfrak{C}$, $\overline{\mathfrak{C}} = \mathfrak{C}^\mathfrak{n}$ and let $\overline{\mathcal{N}^\beta} = (\mathcal{N}^\beta)^\mathfrak{n}$. Recall that $\overline{\mathfrak{C}}$ and $\overline{\mathcal{N}^\beta}$ are structures for the language $\mathcal{L}_\mathfrak{n}$ (see subsection 1.1). The real mouse \mathcal{N}^β is a mouse iterate of its core \mathfrak{C} , and $\overline{\mathcal{N}^\beta}$ is a premouse iterate of $\overline{\mathfrak{C}}$. In particular, Lemma 1.14 implies that the mouse iterate $\mathfrak{C}_\kappa = \mathcal{N}^\beta$ and the premouse iterate $\overline{\mathfrak{C}}_\kappa = \overline{\mathcal{N}^\beta}$. Let $\pi_{0\kappa}: \overline{\mathfrak{C}} \xrightarrow[\Sigma_1]{\quad} \overline{\mathcal{N}^\beta}$ be the premouse embedding of $\overline{\mathfrak{C}}$ into its κ^{th} premouse iterate $\overline{\mathcal{N}^\beta}$.

Claim. $\rho_{\overline{\mathfrak{C}}}^{k+1} = 1$ and $\rho_{\overline{\mathfrak{C}}}^k > 1$ for some $k \geq 0$.

Before we prove this claim we show how it can be used to complete the proof of the theorem. Let $k \in \omega$ be as in the statement of the Claim. Corollary 3.30 of [1] (also see [1, Definition 3.26]) implies that there exists a $\Sigma_{k+1}(\overline{\mathfrak{C}})$ set of reals A not in $\overline{\mathfrak{C}}$. Because the real mouse \mathcal{N}^β is a mouse iterate of its core \mathfrak{C} (see Theorem 2.33 of [4]), Lemma 2.19 of [4] implies that A is a $\Sigma_{\mathfrak{n}+k+1}(\mathcal{N}^\beta)$ set of reals that is not in \mathcal{N}^β . Hence, the desired conclusion follows.

Proof of Claim. Assume, for a contradiction, that $\rho_{\overline{\mathfrak{C}}}^k > 1$ for all $k \geq 1$. Since β properly ends the $\Sigma_1(\mathcal{N})$ -gap, there is a Σ_1 formula $\varphi(v)$ in the language $\mathcal{L} = \{\in, \mathbb{R}, \kappa, \mu\}$ and a real x such that $\mathcal{N}^{\beta+1} \models \varphi(x)$ and $\mathcal{N}^\beta \not\models \varphi(x)$. Thus, there is an $i \in \omega$ such that $(S_{\beta+i}^\mathcal{N}(\mathbb{R}), \mathbb{R}, \kappa, \mu) \models \varphi(x)$ (see subsection 3.1 of [1]). By Proposition 3.85 of [1], there is a formula $\chi(v)$ in the language⁸ \mathcal{L}^μ such that $(\mathcal{N}^\beta, \mu^{\beta+1}) \models \chi(x)$ and $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \not\models \chi(x)$ for all $\gamma < \beta$. Because $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$, Lemma 1.14 asserts that $\mu^{\beta+1}$ is κ_0 -predictable (see Definition 1.11). Therefore, as in the proof of Lemma 3.87 of [1], there is a Σ_ω formula $\psi(u, v)$ in the language \mathcal{L} such that

$$(4.4) \quad (\mathcal{N}^\beta, \mu^{\beta+1}) \models \chi(x) \text{ iff } \mathcal{N}^\beta \models \psi(x, \kappa_0)$$

where ψ depends only on χ and \mathfrak{n} ; that is, the equivalence (4.4) holds for any such real mouse \mathcal{N}^β where κ_0 is the “measurable cardinal” of its core.

It follows from Theorem 3.59 of [1] (also see [4, Theorem 2.11]) that there is a canonical Σ_k formula $\overline{\psi}(u, v)$ in the language $\mathcal{L}_\mathfrak{n}$ such that

$$(4.5) \quad \mathcal{N}^\beta \models \psi(x, \kappa_0) \text{ iff } \overline{\mathcal{N}^\beta} \models \overline{\psi}(x, \kappa_0).$$

Let $f \in \mathfrak{C}$ be the identity function $f: \kappa_0 \rightarrow \kappa_0$. Because $\pi_{0\kappa}(f)(\kappa_0) = \kappa_0$, Lemma 2.19 of [3] implies that there is a Σ_k formula $\overline{\psi}^*(u, v)$ in the language $\mathcal{L}_\mathfrak{n}$, depending only on $\overline{\psi}$, so that

$$(4.6) \quad \overline{\mathcal{N}^\beta} \models \overline{\psi}(x, \kappa_0) \text{ iff } \overline{\mathfrak{C}} \models \overline{\psi}^*(x, f).$$

Lemma 3.71 and its Corollary 3.72 of [1] assert that $\overline{\mathfrak{C}}$ is $(k+1)$ -sound and $\overline{\mathfrak{C}}$ has a Σ_{k+1} Skolem function. Let a be an element of $\overline{\mathfrak{C}}$ so that $\overline{\mathfrak{C}}$ has a Σ_{k+1} Skolem function which is $\Sigma_{k+1}(\{a\})$ (see [1, Definition 3.4]). Let $\mathcal{H} = \text{Hull}_{k+1}^{\overline{\mathfrak{C}}}(\mathbb{R} \cup \{a, f\})$, and let $\overline{\mathcal{K}}$ be the transitive collapse of \mathcal{H} . Let $\overline{\sigma}: \overline{\mathcal{K}} \xrightarrow[\Sigma_{k+1}]{\quad} \overline{\mathfrak{C}}$ be the inverse of the collapse map. As in the proof of Lemma

2.29 of [4], there is a real mouse \mathcal{K} such that $n(\mathcal{K}) = n(\mathfrak{C}) = \mathfrak{n}$ and $\rho_{\mathcal{K}}^{\mathfrak{n}+1} \leq \rho_{\overline{\mathfrak{C}}}^{\mathfrak{n}+1} < \kappa$. Note that \mathcal{K} is a core mouse where $\overline{\sigma}(\kappa_0) = \kappa^\mathcal{K}$ and $\overline{\sigma}(f): \overline{\sigma}(\kappa_0) \rightarrow \overline{\sigma}(\kappa_0)$ is the identity function in $\overline{\mathcal{K}}$. In addition, $\overline{\mathcal{K}} \models \overline{\psi}^*(x, \overline{\sigma}(f))$. Since κ is a regular cardinal greater than Θ , it follows that the mouse iterates \mathfrak{C}_κ and \mathcal{K}_κ are comparable. Due to the fact that $\overline{\sigma}: \overline{\mathcal{K}} \xrightarrow[\Sigma_{k+1}]{\quad} \overline{\mathfrak{C}}$, we conclude

that \mathcal{K}_κ must be an initial segment of \mathfrak{C}_κ . Since $\mathfrak{C}_\kappa = \mathcal{N}^\beta$, there must be an ordinal $\gamma \leq \beta$ such that $\mathcal{K}_\kappa = \mathcal{N}^\gamma$. Thus, \mathcal{N}^γ is a real mouse with core \mathcal{K} . Recall that $\overline{\mathcal{K}} \models \overline{\psi}^*(x, \overline{\sigma}(f))$. Because

⁷See footnote 4.

⁸See footnote 5.

the biconditionals (4.4)–(4.5) are sufficiently uniform, we conclude that $(\mathcal{N}^\gamma, \mu^{\gamma+1}) \models \chi(x)$ and hence, $\mathcal{N}^{\gamma+1} \models \varphi(x)$. Therefore, $\mathcal{N}^{\beta+1} = \mathcal{N}^{\gamma+1}$ and thus, $\mathcal{N}^\beta = \mathcal{N}^\gamma$. Consequently, $\mathfrak{C} = \mathcal{K}$ and $\overline{\mathfrak{C}} = \overline{\mathcal{K}}$. It now follows from the construction of $\overline{\mathcal{K}}$ that there is a $\Sigma_{k+1}(\overline{\mathfrak{C}})$ set of reals not in $\overline{\mathfrak{C}}$. Therefore, $\rho_{\overline{\mathfrak{C}}}^{k+1} = 1$ and this contradiction ends our proof of the Claim. \square

The proof of the theorem is now complete. \square

Remark 4.20. Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap and let $\mathfrak{n} = n(\mathcal{N}^\beta)$. Lemma 3.88(5) of [1] and the above Theorem 4.19 imply that $m(\mathcal{N}^\beta)$ is defined if and only if $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$.

Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap. Theorem 4.19 implies that if $\mathcal{P}(\mathbb{R}) \cap N^{\beta+1} \setminus N^\beta = \emptyset$, then $\mathcal{N}^{\beta+1}$ is “ \mathbb{R} -sharplike”⁹. It follows that \mathbb{R} -sharplike real mice relate to an inner model of $V = K(\mathbb{R})$ in the same way that $\mathbb{R}^\#$ relates to $L(\mathbb{R})$. This, together with Theorem 5.17 of [3], allows one to prove the following theorem.

Theorem 4.21. *Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse. Suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap and let $\mathfrak{n} = n(\mathcal{N}^\beta)$. Assuming $\mathcal{N}^\beta \models \text{AD}$, if $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} = \kappa$ then the pointclasses $\Sigma_n(\mathcal{N}^\beta)$ and $\Pi_n(\mathcal{N}^\beta)$ do not have the scale property for all $n \geq 1$.*

Sketch of Proof. For $\gamma \leq \widehat{\text{OR}}^\mathcal{N}$ let $H_\kappa^{\mathcal{N}^\gamma} = \{a \in N^\gamma : |T_c(a)|_{\mathcal{N}^\gamma} < \kappa\}$ (see [1, Definition 3.18]). Since $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} = \kappa$, Lemma 4.3 of [1] implies that $H_\kappa^{\mathcal{N}^\beta} = H_\kappa^{\mathcal{N}^{\beta+1}}$. One can prove (see [7, Chapter 15]) that the sets of reals in $H_\kappa^{\mathcal{N}^\beta}$ are exactly those in an inner model of $V = K(\mathbb{R})$. Thus, Theorem 5.17 of [3] implies that there is a set of reals in $H_\kappa^{\mathcal{N}^\beta}$ which has no scale in $H_\kappa^{\mathcal{N}^\beta}$. Because $H_\kappa^{\mathcal{N}^\beta} = H_\kappa^{\mathcal{N}^{\beta+1}}$, it follows that $\Sigma_n(\mathcal{N}^\beta) = H_\kappa^{\mathcal{N}^\beta}$ as pointclasses, for each n . Therefore, $\Sigma_n(\mathcal{N}^\beta)$ and $\Pi_n(\mathcal{N}^\beta)$ do not have the scale property. \square

Lemma 4.22. *Let $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ be an iterable real premouse and suppose that β properly ends a $\Sigma_1(\mathcal{N})$ -gap and let $\mathfrak{n} = n(\mathcal{N}^\beta)$. Suppose that $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$ and let $\mathfrak{m} = m(\mathcal{N}^\beta)$. Then*

$$\Sigma_{n+\mathfrak{m}}(\mathcal{N}^\beta) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_{\mathfrak{m}}(\mathcal{N}^\beta)))$$

as pointclasses, for all $n \geq 0$.

Proof. Because $\rho_{\mathcal{N}^\beta}^{\mathfrak{n}+1} < \kappa$, Theorem 4.19 implies that $\mathfrak{m} = m(\mathcal{N}^\beta)$ is defined. Let $\mathfrak{C} = \mathfrak{C}(\mathcal{N}^\beta)$. Because \mathcal{N}^β is a mouse iterate of \mathfrak{C} , we see that $m(\mathfrak{C}) = \mathfrak{m}$ by Lemma 2.19 of [4]. Note that $\rho_{\mathcal{N}^\beta}^0 > 1$ by definition (see [4, Definition 1.18]). We will now show that $\rho_{\mathcal{N}^\beta}^i > 1$ for all $0 \leq i < \mathfrak{m}$. Suppose that for some $i < \mathfrak{m}$ we have that $\rho_{\mathcal{N}^\beta}^i = 1$. Assume that $i \geq 1$ is the smallest such natural number. Thus, $\rho_{\mathcal{N}^\beta}^{i-1} > 1$. Corollary 2.38 of [4] then implies that there is a $\Sigma_i(\mathcal{N}^\beta)$ set of reals not in \mathcal{N}^β , which is impossible because $i < \mathfrak{m}$. We conclude $\rho_{\mathcal{N}^\beta}^{\mathfrak{m}-1} > 1$ and thus, Corollary 2.38 of [4] also implies that $\rho_{\mathfrak{C}}^{\mathfrak{m}-1} > 1$. By Lemma 2.34 of [4] and Corollary 1.32 of [4], there is a partial $\Sigma_{\mathfrak{m}}(\mathfrak{C})$ map $g: \mathbb{R} \xrightarrow{\text{onto}} C$, where C is the domain of \mathfrak{C} . For $n \geq 0$ it follows, as in Proposition 4.6,¹⁰ that $\Sigma_{n+\mathfrak{m}}(\mathfrak{C}) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_{\mathfrak{m}}(\mathfrak{C})))$ as pointclasses. We know that \mathcal{N}^β is a mouse iterate of \mathfrak{C} and thus, $\Sigma_{k+\mathfrak{m}}(\mathfrak{C}) = \Sigma_{k+\mathfrak{m}}(\mathcal{N}^\beta)$ as pointclasses for $k = 0$ and $k = n$, by Lemma 2.19 of [4]. Therefore, $\Sigma_{n+\mathfrak{m}}(\mathcal{N}^\beta) \subseteq \Sigma_n^*(\mathfrak{B}(\Sigma_{\mathfrak{m}}(\mathcal{N}^\beta)))$. \square

Definition 4.23. Let \mathcal{N} be an iterable real premouse and let $[\alpha, \beta]$ be a $\Sigma_1(\mathcal{N})$ -gap. If $\alpha < \beta < \widehat{\text{OR}}^\mathcal{N}$ and $m(\mathcal{N}^\beta)$ is defined, then we shall say that

- β is *weak* if and only if \mathcal{N}^β is a weak mouse,
- β is *strong* if and only if \mathcal{N}^β is a strong mouse.

Our next theorem is an observation that can be used to prove within $K(\mathbb{R})$ that the axiom of determinacy is equivalent to the existence of arbitrarily large cardinals $\kappa < \Theta$ with the strong partition property (see [9, page 432]).

⁹One can generalize Dodd’s theorems (see [7, Chapter 15]) concerning “sharplike mice” to encompass “ \mathbb{R} -sharplike real mice”.

¹⁰See Examples 1 and 2.

Theorem 4.24. *Suppose that $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap where β is weak. Let $\mathfrak{m} = m(\mathcal{N}^\beta)$. If $\mathcal{N}^\alpha \models \text{AD}$, then the pointclass $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)$ has the scale property.*

Proof. Suppose that $\mathcal{N}^\alpha \models \text{AD}$. If $\mathcal{N}^\beta \models \text{AD}$, then Theorem 1.3 asserts that $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)$ has the scale property. We shall prove that $\mathcal{N}^\beta \models \text{AD}$. Assume, for a contradiction, that $\mathcal{N}^\beta \not\models \text{AD}$. Thus, there is a non-determined game in \mathcal{N}^β and this can be asserted as a Σ_1 statement true in \mathcal{N}^β . Since $\mathcal{N}^\alpha \prec_1^{\mathbb{R}} \mathcal{N}^\beta$, there is a non-determined game in \mathcal{N}^α . This contradiction completes the proof of the theorem. \square

Lemma 4.25. *Suppose that $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap where β is weak. Let $\mathfrak{m} = m(\mathcal{N}^\beta)$. Then*

- (a) $\underline{\Sigma}_{n+\mathfrak{m}}(\mathcal{N}^\beta) = \Sigma_n^*(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta))$
- (b) $\underline{\Pi}_{n+\mathfrak{m}}(\mathcal{N}^\beta) = \Pi_n^*(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta))$

as pointclasses, for all $n \geq 0$.

Proof. For $n = 0$, the conclusion holds by Definition 4.4. So we assume $n \geq 1$. It is sufficient to show that $\underline{\Sigma}_{n+\mathfrak{m}}(\mathcal{N}^\beta) = \Sigma_n^*(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta))$. Clearly, $\Sigma_n^*(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)) \subseteq \underline{\Sigma}_{n+\mathfrak{m}}(\mathcal{N}^\beta)$. We show that $\underline{\Sigma}_{n+\mathfrak{m}}(\mathcal{N}^\beta) \subseteq \Sigma_n^*(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta))$. By Lemma 4.22 we have that

$$\underline{\Sigma}_{n+\mathfrak{m}}(\mathcal{N}^\beta) \subseteq \Sigma_n^*(\mathfrak{B}(\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)))$$

and thus it is enough to show, as in the proof of Lemma 4.12, that $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta) \subseteq \exists^{\mathbb{R}} \underline{\Pi}_{\mathfrak{m}}(\mathcal{N}^\beta)$. To do this, let $P \subseteq \mathbb{R}$ be in $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)$. Because β is weak, Theorem 1.5 states that there is a total $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)$ map $k: \omega \rightarrow N^\beta$ such that $P = \bigcup_{i \in \omega} k(i)$ and thus

$$P(x) \text{ iff } (\exists i \in \omega)(\forall a)(a = k(i) \rightarrow x \in a)$$

for all $x \in \mathbb{R}$. Since the graph of k is $\underline{\Sigma}_{\mathfrak{m}}(\mathcal{N}^\beta)$, we conclude that P is in $\exists^{\mathbb{R}} \underline{\Pi}_{\mathfrak{m}}(\mathcal{N}^\beta)$ and so, P is in $\exists^{\mathbb{R}} \underline{\Pi}_{\mathfrak{m}}(\mathcal{N}^\beta)$. \square

Theorem 4.26. *Suppose that $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap and β properly ends this gap. If β is weak, $\mathfrak{m} = m(\mathcal{N}^\beta)$ and $\mathcal{N}^{\beta+1} \models \text{AD}$, then the pointclasses*

$$\underline{\Sigma}_{\mathfrak{m}+2k}(\mathcal{N}^\beta) \text{ and } \underline{\Pi}_{\mathfrak{m}+(2k+1)}(\mathcal{N}^\beta)$$

have the scale property, for all $k \geq 0$.

Proof. This follows directly from Lemma 4.25, Theorem 4.24 and the Second Periodicity Theorem of Moschovakis [11, Theorems 6C.2 and 6C.3] (see Remark 4.5). \square

When β properly begins a $\Sigma_1(\mathcal{N})$ -gap and is strong, then Martin's arguments in [10] also give the following analogues of Theorem 3.3 and Corollary 3.4, respectively, in [12].

Theorem 4.27. *Suppose that $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap and β properly ends this gap. If β is strong and $\mathcal{N}^{\beta+1} \models \text{AD}$, then there is a $\Pi_1(\mathcal{N}^\alpha)$ subset of $\mathbb{R} \times \mathbb{R}$ with no uniformization in $N^{\beta+1}$.*

Corollary 4.28. *Suppose that $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap and β properly ends this gap. If β is strong and $\mathcal{N}^{\beta+1} \models \text{AD}$, then the pointclasses*

$$\underline{\Sigma}_{n+1}(\mathcal{N}^\beta) \text{ and } \underline{\Pi}_n(\mathcal{N}^\beta)$$

do not have the scale property for all $n \geq 1$.

4.4. The scale table. In this subsection we shall assume that the iterable real premouse $\mathcal{N} = (N, \mathbb{R}, \kappa, \mu)$ is such that $\mathcal{N} \models \text{AD}$ and thus, $\Sigma_1(\mathcal{N})$ has the scale property. In the previous subsections we identified precisely those internal levels of the Levy hierarchy for \mathcal{N} which also possess the scale property. Table 1 presents a summary of this development where $[\alpha, \beta]$ is a $\Sigma_1(\mathcal{N})$ -gap, $n \geq 1$, $\mathcal{O} = \widehat{\text{OR}}^{\mathcal{N}}$, $\mathfrak{n} = n(\mathcal{N}^\beta)$ and $\mathfrak{m} = m(\mathcal{N}^\beta)$ whenever $m(\mathcal{N}^\beta)$ is defined. In Table 1, items 3–6 focus on the proper beginning of a $\Sigma_1(\mathcal{N})$ -gap, items 7–8 address the interior of such a gap and items 9–14 concentrate on the proper ending of a gap. When β is not proper, then $\mathcal{N} = \mathcal{N}^\beta$ and Table 1 does not address the question of whether or not the external pointclasses $\underline{\Sigma}_n(\mathcal{N})$ or $\underline{\Pi}_n(\mathcal{N})$ have the scale property for arbitrary n . In Section 7, we shall pursue this issue.

	POINTCLASS	GAP PROPERTY	SCALE PROPERTY
1.	$\underline{\Sigma}_1(\mathcal{N}^\alpha)$	$\alpha \leq \mathcal{O}$	Yes
2.	$\underline{\Pi}_1(\mathcal{N}^\alpha)$	$\alpha \leq \mathcal{O}$	No
3.	$\underline{\Sigma}_n(\mathcal{N}^\alpha)$	α is uncollectible & $\alpha < \mathcal{O}$	Yes iff n is odd
4.	$\underline{\Pi}_n(\mathcal{N}^\alpha)$	α is uncollectible & $\alpha < \mathcal{O}$	Yes iff n is even
5.	$\underline{\Sigma}_{n+1}(\mathcal{N}^\alpha)$	α is collectible & $\alpha < \mathcal{O}$	No
6.	$\underline{\Pi}_n(\mathcal{N}^\alpha)$	α is collectible & $\alpha < \mathcal{O}$	No
7.	$\underline{\Sigma}_n(\mathcal{N}^\gamma)$	$\alpha < \gamma < \beta$	No
8.	$\underline{\Pi}_n(\mathcal{N}^\gamma)$	$\alpha < \gamma < \beta$	No
9.	$\underline{\Sigma}_n(\mathcal{N}^\beta)$	β is weak & $\beta < \mathcal{O}$	Yes iff $(n-\mathbf{m}) \geq 0$ is even
10.	$\underline{\Pi}_n(\mathcal{N}^\beta)$	β is weak & $\beta < \mathcal{O}$	Yes iff $(n-\mathbf{m}) \geq 1$ is odd
11.	$\underline{\Sigma}_n(\mathcal{N}^\beta)$	β is strong & $\beta < \mathcal{O}$	No
12.	$\underline{\Pi}_n(\mathcal{N}^\beta)$	β is strong & $\beta < \mathcal{O}$	No
13.	$\underline{\Sigma}_n(\mathcal{N}^\beta)$	$\rho_{\mathcal{N}^\beta}^{\mathbf{n}} = \kappa$ & $\beta < \mathcal{O}$	No
14.	$\underline{\Pi}_n(\mathcal{N}^\beta)$	$\rho_{\mathcal{N}^\beta}^{\mathbf{n}} = \kappa$ & $\beta < \mathcal{O}$	No

TABLE 1. Scale analysis of a $\Sigma_1(\mathcal{N})$ -gap $[\alpha, \beta]$ where $\mathcal{O} = \widehat{\text{OR}}^\mathcal{N}$, $\mathbf{n} = n(\mathcal{N}^\beta)$ and $\mathbf{m} = m(\mathcal{N}^\beta)$.

This completes the results of Section 4. It turns out that these results will allow us to answer all of our questions concerning the complexity of scales in the inner model $K(\mathbb{R})$ (see Section 6).

5. PREMOUSE ITERATION PRESERVES Σ_1 -GAPS

In this section we shall show the premouse iteration preserves Σ_1 -gaps and preserves internal pointclasses. Then we will show that the iteration of a real 1-mouse also preserves its external pointclasses. Throughout this section \mathcal{N} will be an iterable premouse with premouse iteration

$$\langle \langle \mathcal{N}_\gamma \rangle_{\gamma \in \text{OR}}, \langle \pi_\gamma : \mathcal{N} \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{N}_\gamma \rangle_{\gamma \in \text{OR}} \rangle.$$

Clearly, $\widehat{\text{OR}}^\mathcal{N} \leq \text{OR}^\mathcal{N}$. If $\widehat{\text{OR}}^\mathcal{N} = \text{OR}^\mathcal{N} = \delta$, it will be convenient to extend the domain of each π_γ to include δ by defining $\pi_\gamma(\delta) = \sup \{ \pi_\gamma(\lambda) : \lambda < \delta \}$.

Theorem 5.1. *Let \mathcal{N} be an iterable premouse and let γ be an ordinal. For all $\alpha < \widehat{\text{OR}}^\mathcal{N}$, we have that $\underline{\Sigma}_n(\mathcal{N}^\alpha) = \underline{\Sigma}_n(\mathcal{N}_\gamma^{\pi_\gamma(\alpha)})$ as pointclasses, for all $n \geq 1$.*

Proof. Let $n \geq 1$. Because $\alpha < \widehat{\text{OR}}^\mathcal{N}$ and $\pi_\gamma : \mathcal{N} \xrightarrow[\Sigma_1]{} \mathcal{N}_\gamma$, we conclude that $\pi_\gamma : \mathcal{N}^\alpha \xrightarrow[\Sigma_\omega]{} \mathcal{N}_\gamma^{\pi_\gamma(\alpha)}$. Consequently, we have that $\underline{\Sigma}_n(\mathcal{N}^\alpha) \subseteq \underline{\Sigma}_n(\mathcal{N}_\gamma^{\pi_\gamma(\alpha)})$, as pointclasses. For the other direction, suppose that A is a set of reals in $\underline{\Sigma}_n(\mathcal{N}_\gamma^{\pi_\gamma(\alpha)})$. Let $\varphi(u, v)$ be a Σ_n formula and let c be an element in $\mathcal{N}^{\pi_\gamma(\alpha)}$ so that

$$x \in A \text{ iff } \mathcal{N}_\gamma^{\pi_\gamma(\alpha)} \models \varphi(x, c)$$

for all $x \in \mathbb{R}$. Let $N_\gamma^{\pi_\gamma(\alpha)}$ be the domain of $\mathcal{N}_\gamma^{\pi_\gamma(\alpha)}$. Since A and $N_\gamma^{\pi_\gamma(\alpha)}$ are in \mathcal{N}_γ , it follows that

$$\mathcal{N}_\gamma \models (\exists y \in N_\gamma^{\pi_\gamma(\alpha)})(\forall x \in \mathbb{R})[x \in A \leftrightarrow \mathcal{N}_\gamma^{\pi_\gamma(\alpha)} \models \varphi(x, y)].$$

Because A is also in \mathcal{N} and $\pi_\gamma: \mathcal{N} \xrightarrow[\Sigma_1]{} \mathcal{N}_\gamma$, we see that

$$\mathcal{N} \models (\exists y \in N^\alpha)(\forall x \in \mathbb{R})[x \in A \leftrightarrow \mathcal{N}^\alpha \models \varphi(x, y)]$$

where N^α is the domain of \mathcal{N}^α . Therefore, A is in $\Sigma_n(\mathcal{N}^\alpha)$. \square

Theorem 5.2. *Let \mathcal{N} be an iterable premouse and let $[\alpha, \beta]$ be a $\Sigma_1(\mathcal{N})$ -gap. For each $\gamma \in \text{OR}$, $[\pi_\gamma(\alpha), \pi_\gamma(\beta)]$ is a $\Sigma_1(\mathcal{N}_\gamma)$ -gap. In addition, $\Sigma_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{N}_\gamma^{\pi_\gamma(\alpha)})$ as pointclasses.*

Proof. Let $[\alpha, \beta]$ be a $\Sigma_1(\mathcal{N})$ -gap. Because $\pi_\gamma: \mathcal{N} \xrightarrow[\Sigma_1]{\text{cofinal}} \mathcal{N}_\gamma$ it follows easily that $[\pi_\gamma(\alpha), \pi_\gamma(\beta)]$ is a $\Sigma_1(\mathcal{N}_\gamma)$ -gap. If α is not proper, then Corollary 2.14(2) of [3] implies that $\Sigma_1(\mathcal{N}^\alpha) = \Sigma_1(\mathcal{N}_\gamma^{\pi_\gamma(\alpha)})$ as pointclasses. If α is proper, then Theorem 5.1 implies the desired conclusion. \square

Corollary 5.3. *Let \mathcal{N} be an iterable premouse and let γ be any ordinal.*

- (1) *If α properly begins a $\Sigma_1(\mathcal{N})$ -gap, then $\pi_\gamma(\alpha)$ properly begins a $\Sigma_1(\mathcal{N}_\gamma)$ -gap.*
- (2) *If β properly ends a $\Sigma_1(\mathcal{N})$ -gap, then $\pi_\gamma(\beta)$ properly ends a $\Sigma_1(\mathcal{N}_\gamma)$ -gap.*

Theorem 5.4. *Let \mathcal{N} be an iterable premouse and let γ be an ordinal. Then $\Sigma_n(\mathcal{N}_\gamma) \subseteq \Sigma_n(\mathcal{N})$ as pointclasses, for all $n \geq 1$.*

Proof. Corollary 2.20 of [3] directly implies this theorem. \square

Theorem 5.5. *If \mathcal{M} is a real 1-mouse, then for any premouse iterate \mathcal{M}_γ we have that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{M}_\gamma)$ as pointclasses, for all $n \geq 1$.*

Proof. Let \mathcal{M} be a 1-mouse. We recall the definition of $\mathfrak{C} = \mathfrak{C}(\mathcal{M})$, the core of \mathcal{M} . Let $\mathcal{H} = \text{Hull}_1^{\mathcal{M}}(\mathbb{R} \cup \omega\rho_{\mathcal{M}} \cup \{p_{\mathcal{M}}\})$, and let \mathfrak{C} be the transitive collapse of \mathcal{H} . It follows that \mathfrak{C} is a real premouse and that \mathcal{M} is a premouse iterate of \mathfrak{C} . Let $n \geq 1$.

Claim. $\Sigma_n(\mathfrak{C}) = \Sigma_n(\mathcal{M})$ as pointclasses.

Proof of Claim. Theorem 5.4 implies that $\Sigma_n(\mathcal{M}) \subseteq \Sigma_n(\mathfrak{C})$. Because \mathfrak{C} and \mathcal{H} are isomorphic structures, $\Sigma_n(\mathfrak{C}) = \Sigma_n(\mathcal{H})$. Since \mathcal{H} is Σ_1 definable (in parameters $\omega\rho_{\mathcal{M}}$ and $p_{\mathcal{M}}$) over \mathcal{M} , we have that $\Sigma_n(\mathcal{H}) \subseteq \Sigma_n(\mathcal{M})$. Therefore, $\Sigma_n(\mathfrak{C}) = \Sigma_n(\mathcal{M})$. \square

Let \mathcal{M}_γ be a premouse iterate of \mathcal{M} . Since \mathcal{M}_γ is also a 1-mouse with $\mathfrak{C}(\mathcal{M}_\gamma) = \mathfrak{C}(\mathcal{M})$ (see Lemma 2.37 of [3]), the above Claim implies that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{M}_\gamma)$ as pointclasses. \square

Theorem 5.6. *Let \mathcal{M} be a real 1-mouse. Then there is an ordinal γ such that the premouse iterate \mathcal{M}_γ is a proper initial segment of a real 1-mouse \mathcal{N} .*

Proof. This follows immediately from Theorem 2.43 of [3]. \square

When \mathcal{M} is a real 1-mouse, the next corollary shows that the question of whether or not the external pointclass $\Sigma_n(\mathcal{M})$ has the scale property can be resolved, via Table 1, for each $n \geq 1$.

Corollary 5.7. *Let \mathcal{M} be a real 1-mouse. Then there exists an iterable real premouse \mathcal{N} and an ordinal $\eta < \widehat{\text{OR}}^{\mathcal{N}}$ such that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{N}^\eta)$ as pointclasses for all $n \geq 1$.*

Proof. Let \mathcal{M} be real 1-mouse. The proof of Lemma 5.4 of [3] implies that there is an ordinal γ and a real 1-mouse \mathcal{N} such that the premouse iterate \mathcal{M}_γ is a proper initial segment of \mathcal{N} . Note that $\pi_\gamma: \mathcal{M} \xrightarrow[\Sigma_1]{} \mathcal{M}_\gamma$ preserves Σ_1 -gaps by Theorem 5.2. Let $\delta = \widehat{\text{OR}}^{\mathcal{M}}$ and define $\pi(\delta) = \sup \{ \pi(\gamma) : \gamma < \widehat{\text{OR}}^{\mathcal{M}} \}$. Note that $\pi(\delta) < \widehat{\text{OR}}^{\mathcal{N}}$. It follows that $\mathcal{M}_\gamma = \mathcal{N}^{\pi(\delta)}$. Hence, Theorem 5.5 implies that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{N}^{\pi(\delta)})$ as pointclasses, for all $n \geq 1$. Now, since $\pi(\delta) < \widehat{\text{OR}}^{\mathcal{N}}$, we see that $\eta = \pi(\delta)$ is as desired. \square

Theorem 5.8. *If \mathcal{M} is a real mouse, then for any mouse iterate \mathcal{M}_γ we have that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{M}_\gamma)$ as pointclasses, for all $n \geq 1$.*

Proof. Lemma 2.19 of [4] implies this theorem. \square

Using an argument similar to the one establishing Corollary 5.7 above, Theorem 5.8 allows us to prove our next corollary. Thus, when \mathcal{M} is a real mouse one can also use Table 1 to determine whether or not the external pointclass $\Sigma_n(\mathcal{M})$ has the scale property, for $n \geq 1$.

Corollary 5.9. *Let \mathcal{M} be a real mouse. Then there exists an iterable real premouse \mathcal{N} and an ordinal $\eta < \widehat{\text{OR}}^{\mathcal{N}}$ such that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{N}^\eta)$ as pointclasses for all $n \geq 1$.*

6. SCALES OF MINIMAL COMPLEXITY IN $K(\mathbb{R})$

Since $K(\mathbb{R})$ is the union of real 1-mice, the development in Section 4 induces a natural Levy hierarchy for the sets of reals and the scales in $K(\mathbb{R})$. Before we identify the scales in $K(\mathbb{R})$ of minimal complexity, we first show that in $K(\mathbb{R})$ there is a close connection between the construction of scales and new Σ_1 truths about the reals. Assume that $K(\mathbb{R}) \models \text{AD}$. Given a scale $\langle \leq_i : i \in \omega \rangle$, we shall denote this scale by $\langle \leq_i \rangle$. Now let A be a set of reals in $K(\mathbb{R})$ and suppose that A has a scale $\langle \leq_i \rangle$ in $K(\mathbb{R})$. By Theorem 5.5 of [3], there is a real 1-mouse $\mathcal{N} \in K(\mathbb{R})$ such that $A \in \mathcal{N}$ and $\langle \leq_i \rangle \in \mathcal{N}$. We will show that there is an ordinal γ that begins a $\Sigma_1(\mathcal{N})$ -gap in which A has a $\Sigma_1(\mathcal{N}^\gamma)$ scale. Let η be the least ordinal such that $A \in \mathcal{N}^{\eta+1}$. Let $[\alpha, \beta]$ be the $\Sigma_1(\mathcal{N})$ -gap containing η . Thus, α properly begins this gap. If A is $\Sigma_1(\mathcal{N}^\alpha)$, then Theorem 4.2 asserts that A has a $\Sigma_1(\mathcal{N}^\alpha)$ scale. If A is not $\Sigma_1(\mathcal{N}^\alpha)$, it follows that β must properly end this gap. To see this, suppose that $\mathcal{N}^\beta = \mathcal{N}$ and thus, $A \in \mathcal{N}^\beta$ and $\langle \leq_i \rangle \in \mathcal{N}^\beta$. Hence, by Wadge's Lemma every $\Pi_1(\mathcal{N}^\alpha)$ subset of $\mathbb{R} \times \mathbb{R}$ has a uniformization in $\Sigma_1(\mathcal{N}^\beta)$. This contradicts Theorem 4.16. Consequently, β properly ends this gap and so, $\beta + 1$ begins a new $\Sigma_1(\mathcal{N})$ -gap. Theorem 4.2 implies that A has a scale in $\Sigma_1(\mathcal{N}^{\beta+1})$. Therefore, as mentioned at the beginning of this paper, the construction of scales in $K(\mathbb{R})$ is closely tied to the verification of new Σ_1 truths in $K(\mathbb{R})$ about the reals. As a consequence of the above argument we have the following two theorems.

Theorem 6.1. *Suppose that \mathcal{N} is an iterable real premouse such that $\mathcal{N} \models \text{AD}$. If a set of reals A admits a scale in \mathcal{N} , then A is $\Sigma_1(\mathcal{N}, \mathbb{R})$.*

Theorem 6.2. *Assume $K(\mathbb{R}) \models \text{AD}$. Any set of reals A admits a scale in $K(\mathbb{R})$ if and only if there is a real 1-mouse \mathcal{N} such that $A \in \Sigma_1(\mathcal{N}^\alpha)$ where α begins a $\Sigma_1(\mathcal{N})$ -gap.*

Furthermore, our method of defining scales in $K(\mathbb{R})$ produces scales of minimal complexity, as is established by Theorem 6.5 below. We shall now give a precise definition of the “complexity” of a scale in $K(\mathbb{R})$.

Definition 6.3. Suppose that $\langle \leq_i \rangle$ is a scale in $K(\mathbb{R})$. Let \mathcal{N} be a real 1-mouse such that $\langle \leq_i \rangle \in \mathcal{N}$. Let $\langle \gamma, k \rangle$ be the lexicographically least so that $\langle \leq_i \rangle$ is $\Sigma_k(\mathcal{N}^\gamma)$ or $\Pi_k(\mathcal{N}^\gamma)$, and $\gamma < \widehat{\text{OR}}^{\mathcal{N}}$. We shall say that $\langle \leq_i \rangle$ has \mathcal{N} -complexity $\langle \gamma, k \rangle$. Now let $\langle \leq_i^* \rangle$ be another scale in \mathcal{N} and let $\langle \gamma^*, k^* \rangle$ be the \mathcal{N} -complexity of $\langle \leq_i^* \rangle$. Then we shall say the \mathcal{N} -complexity of $\langle \leq_i \rangle$ is less than or equal to the \mathcal{N} -complexity of $\langle \leq_i^* \rangle$, denoted by $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$, when $\langle \gamma, k \rangle$ is lexicographically less than or equal to $\langle \gamma^*, k^* \rangle$.

Let \mathcal{N} be any real 1-mouse containing the scale $\langle \leq_i \rangle$ as an element. Lemma 4.11 of [1] implies that the \mathcal{N} -complexity of $\langle \leq_i \rangle$ is always defined.

Definition 6.4. Let $\langle \leq_i \rangle$ and $\langle \leq_i^* \rangle$ be scales in $K(\mathbb{R})$. We shall say the complexity of $\langle \leq_i \rangle$ is less than or equal to the complexity of $\langle \leq_i^* \rangle$, denoted by $\langle \leq_i \rangle \preceq \langle \leq_i^* \rangle$, if and only if $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$ for some 1-mouse \mathcal{N} .

We note that the definition of the relation \preceq is independent of the 1-mouse \mathcal{N} (see Section 5) and is a prewellordering on the scales in $K(\mathbb{R})$.

Theorem 6.5. *Assume that $K(\mathbb{R}) \models \text{AD}$. Let $\langle \leq_i^* \rangle$ be a scale in $K(\mathbb{R})$ on a set of reals A . Then there is a scale $\langle \leq_i \rangle$ on A in $K(\mathbb{R})$ constructed as in Section 4 such that $\langle \leq_i \rangle \preceq \langle \leq_i^* \rangle$.*

Proof. Let $\langle \leq_i^* \rangle$ in $K(\mathbb{R})$ be a scale on a set of reals A . Let \mathcal{N} be a real 1-mouse such that $\langle \leq_i^* \rangle \in \mathcal{N}$. Let η be the least ordinal such that $\langle \leq_i^* \rangle \in \mathcal{N}^{\eta+1}$. It follows that $A \in \mathcal{N}^{\eta+1}$. Let $[\alpha, \beta]$ be the $\Sigma_1(\mathcal{N})$ -gap containing η . Thus, α properly begins this gap. If A is in $\Sigma_1(\mathcal{N}^\alpha)$, then Theorem 4.2 asserts that A has a $\Sigma_1(\mathcal{N}^\alpha)$ scale $\langle \leq_i \rangle$. Since $\langle \leq_i^* \rangle \notin \mathcal{N}^\alpha$ it follows that $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$. Thus, if A is in $\Sigma_1(\mathcal{N}^\alpha)$, then the conclusion of the theorem follows. For the remainder of the proof we shall assume that A is not in $\Sigma_1(\mathcal{N}^\alpha)$. We focus on the three cases: (1) $\eta = \alpha$, (2) $\alpha < \eta < \beta$ and (3) $\alpha < \eta = \beta$.

CASE 1: $\alpha = \eta$. Thus, $\langle \leq_i \rangle \in \mathcal{N}^{\alpha+1}$ and $A \in \mathcal{N}^{\alpha+1}$. By Lemma 4.11 of [1], there is a smallest natural number $n \geq 1$ such that $A \in \Sigma_n(\mathcal{N}^\alpha) \cup \Pi_n(\mathcal{N}^\alpha)$.

SUBCASE 1.1: α is uncollectible. Suppose that n is odd. If A is in $\Sigma_n(\mathcal{N}^\alpha)$, then Theorem 4.12 implies that A has a scale $\langle \leq_i \rangle$ in $\Sigma_n(\mathcal{N}^\alpha)$. It follows that $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$. Suppose now that A is not in $\Sigma_n(\mathcal{N}^\alpha)$ and thus, A is in $\Pi_n(\mathcal{N}^\alpha)$. It follows that $\langle \leq_i^* \rangle$ is not in $\Pi_n(\mathcal{N}^\alpha)$. Otherwise, since A is not in $\Sigma_n(\mathcal{N}^\alpha)$, Wadge's Lemma would imply that $\Pi_n(\mathcal{N}^\alpha)$ has the scale property. In this case, however, $\Sigma_n(\mathcal{N}^\alpha)$ has the scale property and thus, $\langle \leq_i^* \rangle$ cannot be in $\Pi_n(\mathcal{N}^\alpha)$ (see 4B.13 of [11]). We conclude that A has a scale $\langle \leq_i \rangle$ in $\Sigma_{n+1}(\mathcal{N}^\alpha)$ and $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$. Similar reasoning applies when n is even.

SUBCASE 1.2: α is collectible. This subcase is not possible, for suppose that α is collectible. Since $\langle \leq_i^* \rangle$ is in $\mathcal{N}^{\alpha+1}$ and A is not in $\Sigma_1(\mathcal{N}^\alpha)$, Wadge's Lemma implies that the every $\Pi_1(\mathcal{N}^\alpha)$ relation has a uniformization in $\mathcal{N}^{\alpha+1}$, contradicting Theorem 4.14.

CASE 2: $\alpha < \eta < \beta$. This case is not possible, for suppose that $\alpha < \eta < \beta$. Since $\langle \leq_i^* \rangle$ is in $\mathcal{N}^{\eta+1}$ and A is not in $\Sigma_1(\mathcal{N}^\alpha)$, Wadge's Lemma implies that the every $\Pi_1(\mathcal{N}^\alpha)$ relation has a uniformization in $\Sigma_1(\mathcal{N}^\beta)$, contradicting Theorem 4.16.

CASE 3: $\alpha < \eta = \beta$. Thus, $\langle \leq_i \rangle \in \mathcal{N}^{\beta+1}$, $A \in \mathcal{N}^{\beta+1}$, $A \notin \mathcal{N}^\beta$ and β properly ends this gap. Theorem 4.18 implies that \mathcal{N}^β is a real mouse and Theorem 4.19 implies that $\mathfrak{m} = m(\mathcal{N}^\beta)$ is defined. By Lemma 4.11 of [1], there is a smallest natural number $n \geq 0$ such that $A \in \Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta) \cup \Pi_{\mathfrak{m}+n}(\mathcal{N}^\beta)$.

SUBCASE 3.1: β is weak. Suppose that n is even. If A is in $\Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta)$, then Theorem 4.26 implies that A has a scale $\langle \leq_i \rangle$ in $\Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta)$. It follows that $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$. Suppose that A is not in $\Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta)$ and thus, A is in $\Pi_{\mathfrak{m}+n}(\mathcal{N}^\beta)$. Thus, $\langle \leq_i^* \rangle$ is not in $\Pi_{\mathfrak{m}+n}(\mathcal{N}^\beta)$. Otherwise, since A is not in $\Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta)$, Wadge's Lemma would imply that $\Pi_{\mathfrak{m}+n}(\mathcal{N}^\beta)$ has the scale property. However, in this case, $\Sigma_{\mathfrak{m}+n}(\mathcal{N}^\beta)$ has the scale property and hence, $\langle \leq_i^* \rangle$ cannot be in $\Pi_{\mathfrak{m}+n}(\mathcal{N}^\beta)$. Consequently, A has a scale $\langle \leq_i \rangle$ in $\Sigma_{\mathfrak{m}+n+1}(\mathcal{N}^\beta)$ and $\langle \leq_i \rangle \preceq_{\mathcal{N}} \langle \leq_i^* \rangle$. Similar reasoning applies when n is odd.

SUBCASE 3.2: β is strong. This subcase is not possible, for suppose that β is strong. Since $\langle \leq_i^* \rangle \in \mathcal{N}^{\beta+1}$, $A \in \mathcal{N}^{\beta+1}$ and A is not in $\Sigma_1(\mathcal{N}^\alpha)$, Wadge's Lemma implies that every $\Pi_1(\mathcal{N}^\alpha)$ relation has a uniformization in $\mathcal{N}^{\beta+1}$, contradicting Theorem 4.27.

This completes the proof. \square

7. POINTCLASS PRESERVING PREMICE

We now direct our attention to the question asked at the beginning of this paper, namely:

Question (Q). Given an iterable real premouse \mathcal{M} and $n \geq 1$, when does the pointclass $\Sigma_n(\mathcal{M})$ have the scale property?

Clearly, if an iterable real premouse \mathcal{M} is a proper initial segment of an iterable real premouse, then the above question can be addressed by referring to Table 1. Suppose now that \mathcal{M} is not a proper initial segment of another iterable real premouse. We know by Corollary 2.14(2) of [3] that the premouse iteration of \mathcal{M} preserves the boldface pointclass $\Sigma_1(\mathcal{M})$, that is, $\Sigma_1(\mathcal{M}_\gamma) = \Sigma_1(\mathcal{M})$ for all ordinals γ . However, if \mathcal{M} is also a real 1-mouse, then Theorem 5.5 asserts that the premouse iteration of \mathcal{M} preserves all of the boldface pointclasses, that is, $\Sigma_n(\mathcal{M}_\gamma) = \Sigma_n(\mathcal{M})$ for all ordinals γ and for all $n \geq 1$. Corollary 5.7 then implies that Question (Q) can be answered. Furthermore, when \mathcal{M} is a real mouse, the fine structure of \mathcal{M} can be used to prove that there is a mouse iterate \mathcal{M}_θ which is a proper initial segment of an iterable real premouse. Theorem 5.8 asserts that mouse iteration preserves the boldface pointclasses. Corollary 5.9 thus implies that the

question as to whether or not the external pointclass $\Sigma_n(\mathcal{M})$ has the scale property can again be addressed.

Let \mathcal{M} be a real mouse which is not a proper initial segment of an iterable real premouse. The above arguments show that one can resolve Question (Q) by utilizing two fundamental attributes of \mathcal{M} :

- (1) \mathcal{M} possesses a specific fine structural property, and
- (2) \mathcal{M} preserves the boldface pointclasses under mouse iteration.

This success inspires a general question. Suppose that \mathcal{M} is merely an iterable real premouse that preserves the boldface pointclasses under premouse iteration. *Can it then be determined which, if any, of its external pointclasses have the scale property?* It may be somewhat surprising to hear that the answer to this question is “yes.” If \mathcal{M} is “pointclass preserving”, we shall show that one can settle Question (Q) without presuming any specific fine structural conditions on \mathcal{M} .

Definition 7.1. Let \mathcal{M} be an iterable real premouse. We say that \mathcal{M} is *pointclass preserving* if, for every premouse iterate \mathcal{M}_γ of \mathcal{M} , we have that $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{M}_\gamma)$ as pointclasses, for each $n \geq 1$.

Remark. When \mathcal{M} is pointclass preserving, then every $\Sigma_n(\mathcal{M})$ set of reals A is also $\Sigma_n(\mathcal{M}_\gamma)$. Definition 7.1 does not assert that the Σ_n formula which defines A over \mathcal{M} is the same formula which defines A over \mathcal{M}_γ . Definition 7.1 only asserts that there is *some* Σ_n definition of A over \mathcal{M}_γ with parameters from \mathcal{M}_γ .

For an iterable real premouse \mathcal{M} , Theorem 5.4 asserts that $\Sigma_n(\mathcal{M}_\gamma) \subseteq \Sigma_n(\mathcal{M})$ (as pointclasses) for any ordinal γ and integer $n \geq 1$. Hence, premouse iteration does not produce any new definable sets of reals. Consequently, \mathcal{M} is pointclass preserving if and only if for each ordinal γ and $n \geq 1$ we have that $\Sigma_n(\mathcal{M}) \subseteq \Sigma_n(\mathcal{M}_\gamma)$. If \mathcal{M} is pointclass preserving, then any premouse iterate of \mathcal{M} does not “lose” any $\Sigma_n(\mathcal{M})$ set of reals. For example, as noted above, real 1–mice are pointclass preserving.

Definition 7.2. Suppose that \mathcal{M} and \mathcal{N} are iterable real premice. Then

- $\mathcal{M} \approx \mathcal{N}$ iff there exists a θ such that $\mathcal{M}_\theta = \mathcal{N}_\theta$,
- $\mathcal{M} \lesssim \mathcal{N}$ iff there exists a θ such that \mathcal{M}_θ is an initial segment of \mathcal{N}_θ ,
- $\mathcal{M} < \mathcal{N}$ iff there exists a θ such that \mathcal{M}_θ is a proper initial segment of \mathcal{N}_θ .

Recall that Definition 1.6 identifies the notion of an \mathbb{R} –complete measure.

Definition 7.3. If ν is an \mathbb{R} –complete measure on κ in $L[\nu](\mathbb{R})$, then $L[\nu](\mathbb{R})$ is said to be a $\rho(\mathbb{R})$ –model with critical point κ .

One can form repeated ultrapowers of a $\rho(\mathbb{R})$ –model $L[\nu](\mathbb{R})$. If each such ultrapower is well-founded, then we say that $L[\nu](\mathbb{R})$ is *iterable*. The next theorem shows that if an iterable real premouse \mathcal{M} is “larger” than all real 1–mice, then there is an iterable $\rho(\mathbb{R})$ –model which contains an iterate of \mathcal{M} as a proper initial segment. If \mathcal{M} is pointclass preserving, then this theorem will allow us to determine if any of the pointclasses $\Sigma_n(\mathcal{M})$ have the scale property.

Theorem 7.4. *Suppose that \mathcal{M} is an iterable real premouse such that $\mathcal{N} \lesssim \mathcal{M}$ for all real 1–mice \mathcal{N} . Then there exists an iterable $\rho(\mathbb{R})$ –model $L[\nu](\mathbb{R})$ with critical point $\lambda > \kappa$ and an ordinal θ such that the premouse iterate \mathcal{M}_θ is an initial segment of $L[\nu](\mathbb{R})$. In addition, $\mathcal{P}(\mathbb{R}) \cap L[\nu](\mathbb{R}) \subseteq K(\mathbb{R})$.*

Proof. Let $\mathcal{M} = (M, \mathbb{R}, \kappa, \mu)$. It follows from Theorem 5.6 that $\mathcal{N} < \mathcal{M}$ for all real 1–mice \mathcal{N} . The proof of Lemma 5.4 of [3] can be used to show that every subset of κ in $K(\mathbb{R})$ is also in some iterate of \mathcal{M} . Hence, Lemma 2.11 of [3] implies that every such subset of κ is in \mathcal{M} . A similar argument shows that any κ –sequence of subsets of κ in $K(\mathbb{R})$ is also in \mathcal{M} . Therefore, μ is a $K(\mathbb{R})$ –measure on $\kappa > \Theta^{K(\mathbb{R})}$ (see Definition 2.1 of [5]). Without loss of generality, one can assume that the measure μ is countably complete.¹¹ By DC and countable completeness, it follows that

¹¹If not, then apply the argument to \mathcal{M}_λ where λ is a sufficiently large regular cardinal.

the ultrapower ${}^\kappa K(\mathbb{R})/\mu$ is well-founded. The proof of Corollary 2.14 of [5] implies that $(K(\mathbb{R}), \mu)$ is really good on κ (see [5, Definition 2.9]). Again, by DC and countable completeness, $(K(\mathbb{R}), \mu)$ is weakly iterable (see [5, Section 2]). Lemma 4.3 of [5] now implies the existence of the desired $\rho(\mathbb{R})$ -model $L[\nu](\mathbb{R})$. \square

Remark 7.5. Let \mathcal{M} and $L[\nu](\mathbb{R})$ be as in the statement of Theorem 7.4. If $K(\mathbb{R}) \models \text{AD}$, then it follows that $L[\nu](\mathbb{R}) \models \text{AD}$ and Lemma 2.11 of [3] implies that $\mathcal{M} \models \text{AD}$.

Corollary 7.6. *Let \mathcal{M} be a pointclass preserving premouse such that $\mathcal{N} \lesssim \mathcal{M}$ for all real 1-mice \mathcal{N} . For all $n \geq 1$ and all sets of reals A , if A is $\Sigma_n(\mathcal{M})$ then $A \in \mathcal{M}$.*

Proof. Let $n \geq 1$ and suppose that A is a set of reals in $\Sigma_n(\mathcal{M})$. Let θ be as in Theorem 7.4. Since \mathcal{M} is pointclass preserving, we have that $\Sigma_n(\mathcal{M}_\theta) = \Sigma_n(\mathcal{M})$ as pointclasses. Theorem 7.4 implies that $A \in K(\mathbb{R})$. Since $\mathcal{N} \lesssim \mathcal{M}$ for all real 1-mice \mathcal{N} , it follows that $\mathcal{N} < \mathcal{M}$ for all real 1-mice \mathcal{N} . Thus, A must be an element of \mathcal{M} . \square

Corollary 7.7. *Assume that $K(\mathbb{R}) \models \text{AD}$. Let \mathcal{M} be a pointclass preserving premouse. Suppose that $\mathcal{N} \lesssim \mathcal{M}$ for all real 1-mice \mathcal{N} . Then,*

- (1) $\Sigma_1(\mathcal{M}, \mathbb{R})$ has the scale property, and
- (2) $\Sigma_n(\mathcal{M})$ and $\Pi_n(\mathcal{M})$ do not have the scale property for any $n \geq 1$.

Proof. Theorem 7.4 implies that $\mathcal{M} \models \text{AD}$ and thus, $\Sigma_1(\mathcal{M}, \mathbb{R})$ has the scale property by Corollary 1.2. Theorem 5.17 of [3] implies that there is a set of reals A in $K(\mathbb{R})$ which has no scale in $K(\mathbb{R})$. Theorem 7.4 implies that $A \in \mathcal{M}$. Using A as a constant, it follows that $\Sigma_n(\mathcal{M})$ does not have the scale property for any $n \geq 1$. \square

Theorem 7.8. *Assume that $K(\mathbb{R}) \models \text{AD}$. Let \mathcal{M} be a pointclass preserving premouse. Suppose that $\mathcal{M} < \mathcal{N}$ for some real 1-mice \mathcal{N} . Then, for $n \geq 1$ one can determine whether or not the pointclass $\Sigma_n(\mathcal{M})$, or $\Pi_n(\mathcal{M})$, has the scale property.*

Proof. Suppose that \mathcal{M} is a pointclass preserving premouse such that $\mathcal{M} < \mathcal{N}$ for some real 1-mice \mathcal{N} . Let θ be so large that \mathcal{M}_θ is an initial segment of \mathcal{N}_θ . Thus, $\Sigma_n(\mathcal{M}) = \Sigma_n(\mathcal{M}_\theta)$ as pointclasses. Since \mathcal{M}_θ is a proper initial segment of \mathcal{N}_θ , let $\gamma < \widehat{\text{OR}}^{\mathcal{N}_\theta}$ be such that $\mathcal{M}_\theta = \mathcal{N}_\theta^\gamma$. Let $[\alpha, \beta]$ be the $\Sigma_1(\mathcal{N}_\theta)$ -gap containing γ . Now Table 1 can be used to determine whether or not $\Sigma_n(\mathcal{M})$ has the scale property. \square

REFERENCES

- [1] Daniel W. Cunningham, *Scales and the fine structure of $K(\mathbb{R})$. Part I: Acceptability above the reals*, Mathematics ArXiv, arXiv:math.LO/0605445.
- [2] ———, *Scales and the fine structure of $K(\mathbb{R})$. Part II: Acceptability above the reals*, Mathematics ArXiv, arXiv:math.LO/0605448.
- [3] ———, *The real core model and its scales*, Ann. Pure Appl. Logic **72** (1995), no. 3, 213–289.
- [4] ———, *The fine structure of real mice*, J. Symbolic Logic **63** (1998), no. 3, 937–994.
- [5] ———, *Is there a set of reals not in $K(\mathbb{R})$?*, Ann. Pure Appl. Logic **92** (1998), no. 2, 161–210.
- [6] A. Dodd and R. Jensen, *The core model*, Ann. Math. Logic **20** (1981), no. 1, 43–75.
- [7] A. J. Dodd, *The core model*, London Mathematical Society Lecture Note Series, vol. 61, Cambridge University Press, Cambridge, 1982.
- [8] R. Björn Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308.
- [9] Akihiro Kanamori, *The higher infinite*, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1994, Large cardinals in set theory from their beginnings.
- [10] Donald A. Martin, *The largest countable this, that, and the other*, Cabal seminar 79–81, Lecture Notes in Math., vol. 1019, Springer, Berlin, 1983, pp. 97–106.
- [11] Yiannis N. Moschovakis, *Descriptive set theory*, Studies in Logic and the Foundations of Mathematics, vol. 100, North-Holland Publishing Co., Amsterdam, 1980.
- [12] John R. Steel, *Scales in $L(\mathbb{R})$* , Cabal seminar 79–81, Lecture Notes in Math., vol. 1019, Springer, Berlin, 1983, pp. 107–156.

MATHEMATICS DEPARTMENT, STATE UNIVERSITY OF NEW YORK, COLLEGE AT BUFFALO, 1300 ELMWOOD AVENUE, BUFFALO, NY 14222, USA

E-mail address: cunnindw@math.buffalostate.edu